BIBLIOTECA NAZIONALE VITT. EM. III NAPOLI
B. Prov.
coll.
4
4

BIBLIOTECA PROVINCIALE
Armadio XXIX
Palchetto 76
Num.° d'ordine 27
4378

~~P4 8 4~~

B. Prov.
Coll 2/(14)

LIBRARY OF USEFUL KNOWLEDGE.

BIBL. VITT. EMANUELE III NAPOLI

PRACTICAL GEOMETRY,

LINEAR PERSPECTIVE,

AND

PROJECTION;

INCLUDING

ISOMETRICAL PERSPECTIVE, PROJECTIONS OF THE
SPHERE, AND THE PROJECTION OF SHADOWS,

WITH

DESCRIPTIONS OF THE PRINCIPAL INSTRUMENTS USED
IN GEOMETRICAL DRAWING, &c.

ILLUSTRATED BY EIGHT PLATES,
AND NUMEROUS WOOD-CUTS.

FOR THE USE OF ARTISTS, ARCHITECTS, ENGRAVERS,
ENGINEERS, MECHANICS, &c.

BY THOMAS BRADLEY.

PUBLISHED UNDER THE SUPERINTENDENCE OF THE SOCIETY FOR THE
DIFFUSION OF USEFUL KNOWLEDGE.

LONDON:
BALDWIN AND CRADOCK, PATERNOSTER-ROW.
MDCCCXXXIV.

VA1 1523 616

LONDON:
Printed by William Clowes,
Duke-street, Lambeth.

ANALYTICAL TABLE OF CONTENTS.

Page

INTRODUCTION 1

Distinction between theoretical and practical geometry . 1

Explanation of the references 3

Description of the instruments made use of in geometrical drawing . 4

- Compasses of different kinds . . 4, 6
  - „ Beam 6
  - „ proportional . . . 7
- Rules 8
  - „ mode of verifying . . . 8, 10
- T square 8
  - „ mode of verifying
- Bevel, or moveable T square . . . 9
- Parallel rules 10
- Scales 11
  - „ Explanation of lines on . . . 11
- Sector 13
- Protractors 13
  - „ paper, employed in laying down surveys . 14
- Sergeant Doul's instrument for *plotting* . 15
- Drawing pen 15

Method of straining paper on drawing boards . . 16

General rules to be observed in all geometrical drawing . . 18

PROBLEMS RELATING TO RIGHT LINES AND ANGLES 19

1. *To draw a perpendicular to a given right line* . 19
   - Method of doing it by the compasses alone . 20
   - „ employed by builders in setting out the angles of an edifice . . . 21
2. *To draw a right line parallel to a given right line* . 21
   - Method of doing it by a rule, or string, alone, without the description of arcs . . 22
3. *To construct an angle of a given magnitude* . 23
   - Method of constructing several angles at a given point by means of a table of natural sines . . 24
4. *To divide a given right line into any required number of equal parts* . . . 25

Page

Method of dividing a line, as employed in diagonal scales 27

Problem 5. *To bisect a given angle* . . . 28

6. *To divide a given right line medially, or in extreme and mean ratio* . . . 29

Method, by the compasses alone . . 30

7. *To find a mean proportional between two given right lines* 30

8. *To find a third proportional to two given right lines* . 30

Method, by the compasses alone . . 31

9. *To find a fourth proportional to three given lines* . 31

Method, by the compasses alone . . 31

Application, for finding the sum of the sides of a rectangle, the area of which shall be in continued proportion to the areas of three given rectangles . 32

10. *To divide a given right line into the same number of segments, and which shall be to each other in the same ratio, as those into which another line is divided* . . . 32

11. *Any number of right lines being given, to find an equal number in the same ratio to each other respectively as those given ones; and having a given ratio to the first* . . . 33

12. *To divide a given right line, either internally or externally, in a given ratio* . . 34

Method, by the proportional compasses . . 35

13. *To divide a given right line, either internally or externally, so that the rectangle under its segments shall be equal to a given rectangle* . 35

14. *To find two lines in a given ratio, which shall contain a rectangle equal to a given one* . . 36

15. *To find an harmonical mean between two given right lines* 37

An easy construction by which the *arithmetical, geometrical, and harmonical* means are found at once 38

16. *To divide a given line in harmonical proportion, and in a given ratio* . . . 38

17. *From two given points, to draw lines to meet in some point of a given right line, so that they shall make equal angles with it* . . 39

18. *To draw through a given point, a line which shall make equal angles with the legs of a given angle* . 39

19. *Through a given point, to draw a right line so that the segments intercepted by the legs of a given angle may be in a given ratio* . . . 40

20. *Four lines meeting in a point being given, to draw a line to cut them all, so that the segments intercepted by the exterior pair shall be equal* . 40

Page

Problem 21. *Given two lines converging to an inaccessible point; to draw a line through a given point, which would, if produced, pass through the inaccessible point* . 41

Application in perspective drawings, to cornices, &c. .

Description of Nicholson's centrolinead . . 42

" Farey's " . . 43

PROBLEMS RELATING TO RIGHT LINE FIGURES . 44

Problem 22. *To construct a triangle from three or more parts given* . 44

23. *To construct a triangle equal to a given rectilinear figure* 45

24. *To construct a triangle equal to a given rectilinear figure, and similar to a given triangle* . . 47

25. *On a side, or a side produced, of a given triangle, to construct another equal to it, which shall have a given point for its vertex* . . 47

26. *On a given line for a base, to construct a triangle equal to a given triangle* . . . 48

27. *On a given line, to construct an isosceles triangle having each angle at the base double that at the vertex* . . . 48

28. *To find the sides of the squares, the areas of which shall be any part or multiple, from 1 to 10, of that of a square on a given line* . . . 48

29. *To construct a parallelogram, equal to a given rectilinear figure* . . . 50

30. *On a given line to describe a parallelogram equal to a given rectilinear figure* . . . 50

31. *To find the side of a square equal in area to any number of given squares added together, or to the difference of two given squares* . . 51

32. *To inscribe a square in a given triangle, so that one side of the square shall coincide with, or form part of, one side of the triangle* . . 51

33. *To divide a given rectilinear figure into any number of equal parts, by lines drawn through a given point* 52

34. *To construct a figure similar to a given figure* . 55

Method of constructing a parallelogram similar to a given one, and having its area in a given ratio to that of this given one . . . 56

Use of the proportional compasses in solving this problem 56

Method employed by artists, &c., to copy irregular figures, pictures, &c. . . . 56

Description of the common pentagraph . . 59

" Wallace's eidograph . 59

35. *To construct regular polygons* . . . 60

Table to facilitate the construction of regular polygons 64

Page

PROBLEMS RELATING TO THE CIRCLE . . 66

Problem 36. *To divide the circumference of a given circle into any number of equal parts* . . . 66

37. *To describe a circle that shall have its circumference equal to a given right line, or its area, equal to a given figure* . . . 67

Method employed by carpenters, &c., for finding the length of a given arc nearly . . 68

38. *To describe a circle to pass through three given points* 68

Method of doing it by means of rules fixed together 68

Method of finding points which would lie in the circumference of the circle passing through three given points 69

39. *Through two given points to describe an arc of a circle that shall contain a given angle* . . 69

Application of the construction in coast surveying . 70

40. *To draw a tangent to a circle through a given point* 70

41. *To draw a line through a given point, so that it shall be cut by the circumference of a given circle in a given ratio* 71

42. *Through two given points to describe a circle, which shall divide the circumference of another given circle in two equal parts* . . . 72

43. *Through a given point to draw a line, so that the segment intercepted by the circumference of a given circle may be equal to a given line* . 72

44. *In, or about, a given circle to construct a triangle similar to a given one* . . . 72

45. *From two given points in the circumference of a given circle, to draw two lines to meet in a third point in it, which shall be to each other in a given ratio* . 73

46. *In a given circle to describe a rectangular parallelogram, equal to a given rectilinear figure* . 73

47. *To divide the area of a given circle into any number of equal parts, (by circles or arcs of circles)* . 73

Properties of the areas of circles and lunes . . 74

48. *Given two circles, not concentric, to draw a third to touch them both* 74

Table of factors to facilitate constructions relating to the circle 75

CONIC SECTIONS 76

Definitions 77

Method of producing these curves by the shadow of a ring . 77

PROBLEMS RELATING TO THE ELLIPSE . 78

" " HYPERBOLA . . 100

" " PARABOLA . 104

Page
Method of describing the *ellipse* by two pins and a thread . . 78
" " *hyperbola* by a pin and a thread . . 103
" " *parabola* " " . . 109
" " " by means of tangents . . 110
Problem 49. *Two conjugate diameters being given by two lines, the curve not being given; to find the axes* of the ellipse . . 79
64. *Do.* *do.* of the hyperbola . . 102
50, 51. *Given the axes, to determine the points where the curve would cut a given line:* of the ellipse . 80, 81
66. *Do.* *do.* of the parabola . 105
52. *A line being given as a diameter, and another bisected by the first at any angle, as a double ordinate; to find the parameter to that diameter* of the ellipse . 82
53. *Given the axes, the foci, and the directrix; to draw lines from a given point which would be tangents to the curve* of an ellipse . . . 83
67. *Do.* of a parabola . . . 106
54. *Given two conjugate diameters to draw from a given point, lines which would be tangents to the curve* of an ellipse . . . 83
55. *Given a line as a diameter of an ellipse, and another line as an ordinate to it; to find the conjugate diameter* of an ellipse . . . 84
56. *To find any number of points which would be in the circumference* of an ellipse . . 84
62. *Do.* of an hyperbola . . 100
65. *Do.* of a parabola . . 104
57. *Given* an ellipse *to find the axes, foci, and directrix* . 86
68. " a parabola *do.* *do.* . 107
58. *Given* an ellipse *to draw any two conjugate diameters; and to draw an ordinate to either through a given point* 87
Method of drawing two conjugate diameters to contain a given angle 88
69. *Given* a parabola *to draw an ordinate to a given diameter, through a given point* . . 107
59. *Given* an ellipse, *to draw a tangent to the curve through a given point* 88
70. *Given* a parabola, *to draw a tangent to the curve through a given point* . . . 107
60. *To find the axes of an ellipse that would be inscribed in a given quadrilateral* . . . 90
61. *To find a rectilinear figure, or a circle equal in area to a given ellipse* . . . 91
71. *Do.* *of a parabola cut off by an ordinate* . 109

Pag

Problem 63. *Given a diameter to find the conjugate and the asymptotes of an hyperbola* . . 101

Properties of the ellipse of occasional use in practice . 91

Description of the trammel . . . 92

" Clement's elliptograph . 93

" Farey's elliptograph . . 94

" a mode of finding points in an ellipse mechanically . . 96

" a simple instrument for drawing a curve resembling an ellipse . . 96

" methods of drawing ellipses by means of rules put together . 97

Construction of a figure resembling an ellipse, composed of arcs of circles . 97

" a figure resembling a parabola composed of arcs of circles . . 109

72. *Given three points to find the axes and the directrices of the conic section which will pass through them, a fourth point for a focus being also given* . 111

THE CYCLOID AND EPICYCLOID . . 112

Mode of describing these curves by finding points in them 112, 115

Properties of the curves of practical use . . 115

Mode of drawing an epicycloid by means of an orrery . 116

THE CONCHOID, and mode of describing by points . 116

Employment of the curve in architecture . . 117

THE CISSOID and mode of describing by points and mechanically . 117

THE SPIRAL—Archimedes—Hyperbolic—Logaritbmic . 118

LINEAR PERSPECTIVE . . 121

INTRODUCTION 123

DEFINITIONS 125, 134

General principles relating to the vanishing points of original lines 125—134

Illustrations of them, by the appearances of streets, &c. . 128, 136

Explanation of the apparent anomaly of parallel lines appearing curved 139

Explanation of the reason, why the columns of a colonnade are projected into larger figures the farther they are from the spectator, under certain circumstances . . 140

OF THE PERSPECTIVE PROJECTION OF RIGHT LINE FIGURES 141

Practical mode of applying the general principles . . 141

To find the vanishing line of a given original plane, inclined in any angle to the plane of projection . . 143

Page

To draw the projection of a given triangle situated in any such given plane 144

Theorem for finding the intersection of the ray from an original point, with the projection of a line in which that point lies; when both are brought into one plane . . 145

Illustration of the principles by a triangle marked out on the ground 147

To draw the projection of a series of squares, like a marble pavement, or a chess-board . . . 149

Theorems of the proportions between original right lines and their projections 150

To draw the projection of a parallelogram, without making use of the vanishing points; by means of the vertical plane and verticals 153

Converse of the foregoing problems, or the mode of finding the original parallelogram, of which a given quadrilateral shall be the projection 155

To draw the projection of a regular polygon . . 156

OF THE VANISHING LINES OF ORIGINAL PLANES . 159

I. When one original plane is parallel to another . 159

II. When one original plane is perpendicular to another . 159

III. When one original plane is oblique to the other . 161

Mode of drawing the projection of a plane solid, by means of the projections of its plans on co-ordinate planes, by which the complex constructions for finding the vanishing lines of its planes are dispensed with . . . 165

ON THE PERSPECTIVE PROJECTIONS OF GEOMETRICAL SOLIDS 167

To draw the projection of a tetrahedron . . 167

" of a cube . . . 172

" of an octohedron by means of the circumscribing cube . . . 173

Remarks on the projections of solids; and on the position of the plane of projection, so as to obtain an agreeable view of an object, such as a building . . . 173

Remarks on the change in the horizontal line produced by changing the elevation of the spectator's eye above the ground 176

Remarks on the mode by which the original building is projected by means of an imaginary model of it . . 179

Application of the principles to drawing a house in perspective 180

Do. do. by means of the plan on a separate drawing board, as applicable to large edifices . . 184

Method of dividing a perspective line into segments representing equal parts of the original . . . 187

Remarks on the errors committed by artists when inserting figures in a perspective drawing . . . 189

Page
Remarks on what is termed a bird's-eye view . 190
To draw a cornice with modillions and pediment in perspective 192
OF THE PERSPECTIVE PROJECTION OF THE CIRCLE AND OTHER CURVES 196
Method of finding the originals of the axes, &c., of the conic sections representing a circle . . 196—204
Remarks on the inclination of the axes of the elliptic projection of a circle, according to changes in the relative position of the eye and the circle . . . 198
Do. do. when the ellipses represent concentric circles 199
Do. do. when the ellipses represent the parallel sections of a cylinder 200
Errors committed by artists in this respect . . 202
Theorems on the conic sections, as representing a circle or other conic section, under different conditions of situation, with respect to the vertex, or eye . . . 204
ON THE PERSPECTIVE PROJECTIONS OF SOLIDS WITH CURVED SURFACES—THE CYLINDER AND CONE . 207
To draw the projection of a circle or other curve by means of the projections of a sufficient number of points lying in it . 210
To draw the projection of two bevelled cog-wheels . 211
Method of dividing the elliptic projection of a circle into segments representing equal arcs . . . 213
To draw the projections of cylinders right or oblique . 216
Method of finding the points in the elliptic ends of a cylinder projected, in which the straight outline of the sides touch them 218
To draw the projection of the curve produced by the intersection of two cylinders 218
To draw the interior of a hall and staircase, with a groined vault 220
Remarks on the projections of descending planes, and the misapprehensions on this point . . . 225
Properties of the solid called an annulus, or ring . . 226
To draw the projection of an annulus . . 228
ORTHOGRAPHIC PROJECTION . . . 230
General principles of 230
To draw the orthographic projection of right line figures . 231
Do. do. plane solids . 233
On the orthographic projection of curves . . 237
ISOMETRICAL PROJECTION . . .
General principles 239
PROJECTIONS OF THE SPHERE, AND OF ITS CIRCLES 243
Stereographic projection of the sphere . . . 243
Globular projection of the sphere . . . 251
Orthographic projection of the sphere . . . 254

Page
Gnomonic projection of the sphere . . . 256
Construction of maps of portions of the earth, by the principles of development 258
Geometrical construction for an approximation to Mercator's projection 260
Geometrical constructions for the development of the spherical surface 260
Perspective projection of the outline of a sphere, considered as an original object 262
THE PROJECTION OF SHADOWS . . . 265
General principles 265
Theorems relating to vanishing lines as connected with the projection of shadows 266
The projection of the shadows of plane solids when these are orthographically projected 271
The projection of the shadows of plane solids when perspectively projected 276
The shadows of curves 279
To draw the shadow of a cylinder, right or oblique; and to find the lines separating the enlightened part from that in shade . 279
To draw the outline of the shadow cast by the edge of the base of a cylinder, on the opposite concave surface . . 283
This outline is an ellipse—demonstrated . . . 284
Method of obtaining the outline of the shadow cast by any curved surface on a plane, or on another curved surface . 285
Example of an annulus 287
To draw the shadow of a sphere cast on a plane . . 287
The shadow of the edge of a hemisphere, cast on the opposite concave spherical surface, is an ellipse . . 290
NOTES AND ILLUSTRATIONS TO THE PLANE GEOMETRY 291

ERRATA.

Page 33, line 11, *for* "from which lines," *read* "from which point draw lines."
„ 64, head of the last column of table; *for* "multiplication of areas," *read* "multipliers for areas."
„ 79, In the enunciation of Prob. 49, *for* "axis," *read* "axes."
„ 80, line 12, the reference ought to be (Note FF.)
„ 83, „ 2 from bottom, *for* "L Q," *read* "C Q."

COMMITTEE.

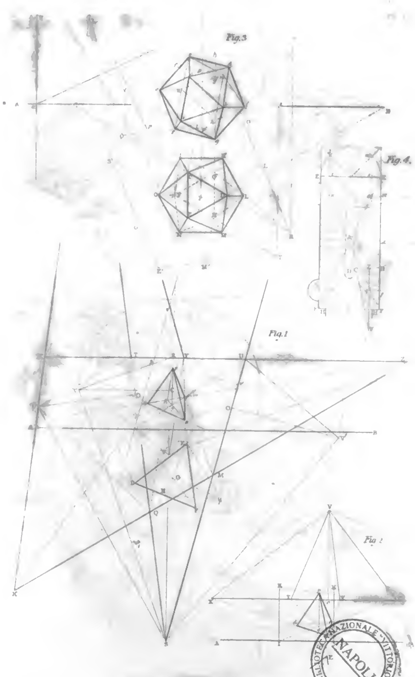
Chairman—The Right Hon. the LORD CHANCELLOR, F.R.S., Memb. Nat. Inst. of France.
Vice-Chairman—The Right Hon. Lord J. RUSSELL, M.P., Paymaster of the Forces.
Treasurer—WILLIAM TOOKE, Esq., M.P., F.R.S.

W. Allen, Esq., F.R. & R.A.S.
Rt. Hon. Visc. Althorp, M.P. Chancellor of the Exch.
Capt. F. Beaufort, R.N., F.R., and R.A.S., Hydrographer to the Admiralty.
Sir C. Bell, F.R.S.L. & E.
G. Burrows, M.D.
C. Hay Cameron, Esq.
The Rt. Rev. the Bishop of Chichester, D.D.
William Coulson, Esq.
R. D. Craig, Esq.
Wm. Crawford, Esq.
J. Fred. Daniell, Esq., F.R.S.
H. T. Delabeche, Esq., V.P. Geol. Soc.
The Right Hon. Lord Denman.
T. Drummond, Esq., R.E., F.R.A.S.
Right Hon. Viscount Ebrington, M.P.
C. L. Eastlake, Esq., R.A.
T. F. Ellis, Esq., A.M., F.R.A.S.
Sir Henry Ellis, F.R.S., Prin. Lib. Brit. Mus.
John Elliotson, M.D., F.R.S.
Thomas Falconer, Esq.
I. L. Goldsmid, Esq., F.R. and R.A.S.
B. Gompertz, Esq., F.R. and R.A.S.
G. B. Greenough, Esq., F.R. & L.S.
H. Hallam, Esq., F.R.S., M.A.
M. D. Hill, Esq., M.P.
Rowland Hill, Esq., F.R.A.S.
Edwin Hill, Esq.
The Right Hon. Sir J. C. Hobhouse, Bart. M.P.
David Jardine, Esq., M.A.
Henry B. Ker, Esq.
The Rt. Hon. the Earl of Kerry
Thos. Hewitt Key, Esq., M.A.
George C. Lewis, Esq., M.A.
James Loch, Esq., M.P., F.G.S.
George Long, Esq., M.A.
J. W. Lubbock, Esq., F.R., R.A., and L.S.S.
H. Malden, Esq., M.A.
A. T. Malkin, Esq. M.A.
James Manning, Esq.
J. Herman Merivale, Esq. M.A., F.A.S.
James Mill, Esq.
W. H. Ord, Esq. M.P.
The Right Hon. Sir H. Parnell, Bart., M.P.
Dr. Roget, Sec. R.S., F.R.A.S.
Sir M. A. Shee, F.R.A., F.R.S.
J. Abel Smith, Esq., M.P.
John Taylor, Esq., F.R.S.
Dr. A. T. Thomson, F.L.S.
H. Waymouth, Esq.
J. Whishaw, Esq., M.A., F.R.S.
John Wood, Esq.
John Wrottesley, Esq., M.A., F.R.A.S.

THOMAS COATES, Esq., Secretary, 59, Lincoln's Inn Fields.

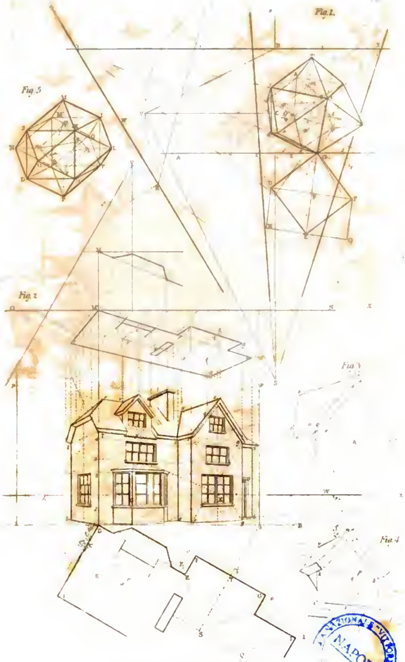
LOCAL COMMITTEES.

Anglesea—Rev. E. Williams.
Rev. W. Johnson.
Mr. Miller.
Ashburton—J. F. Kingston, Esq.
Barnstaple— — Bancraft, Esq.
William Gribble, Esq.
Bilston—Rev. W. Leigh.
Birmingham—Rev. John Corrie, F.R.S., *Chairman*.
Paul M. James, Esq., *Treasurer*.
Bridport—Wm. Forster, Esq.
James Williams, Esq.
Bristol—J. N. Sanders, Esq., *Chairman*.
J. Reynolds, Esq., *Treas.*
J. B. Estlin, Esq., F.L.S., *Sec.*
Calcutta—Lord Wm. Bentinck.
Sir Edward Ryan.
James Young, Esq.
Cambridge—Rev. James Bowstead, M.A.
Rev. Prof. Henslow, M.A., F.L.S. & G.S.
Rev. Leonard Jenyns, M.A., F.L.S.
Rev. John Lodge, M.A.
R. W. Rothman, Esq., M.A., F.R.A.S. and G.S.
Rev. Geo. Peacock, M.A., F.R.S. & G.S.
Rev. Prof. Sedgwick, M.A., F.R.S. & G.S.
Professor Smyth, M.A.
Rev. C. Thirlwall, M.A.
Canterbury—Alex. B. Higgins, Esq.
John Brent, Esq.
Dr. Harry Wm. Carter, M.D., F.R.S.E.
William Masters, Esq.
Canton—J. F. Davis, Esq., F.R.S.
Cardigan—The Rev. J. Blackwell.
Carnarvon—R. A. Poole, Esq.
William Roberts, Esq.
Chester—Hayes Lyon, Esq.
Henry Potts, Esq.
Chichester—Dr. Forbes, F.R.S.
C. C. Dendy, Esq.
Coventry—Ar. Gregory, Esq.
Corfu—John Crawford, Esq.
Mr. Plato Petrides.
Denbigh—John Madocks, Esq.
Thos. Evans, Esq.
Derby—Joseph Strutt, Esq.
Edward Strutt, Esq., M.P.
Devonport and Stonehouse—
John Cole, Esq.
— Norman, Esq.
Lt.-Col. C. Hamilton Smith, F.R.S.
Etruria—Jos. Wedgwood, Esq.
Exeter—J. Tyrrell, Esq.
John Milford, Esq. (Coaner.)
Glasgow—K. Finlay, Esq.
Professor Mylne.
Alexander McGrigor, Esq.
Charles Tennant, Esq.
James Cowper, Esq.
Glamorganshire—
Dr. Malkin, Cowbridge
W. Williams, Esq., Aberpergwm.
Guernsey—F. C. Lukis, Esq.
Hull—J. C. Parker, Esq.
Keighley, Yorkshire—Rev. T. Dury, M.A.
Launceston—Rev. J. Barfitt.
Leamington Spa—Dr. Loudon, M.D.
Leeds—J. Marshall, Esq.
Lewes—J. W. Woollgar, Esq.
Liverpool Local Association—
W. W. Currie, Esq., *Chairman*.
J. Mulleneux, Esq., *Treas.*
Rev. W. Shepherd
J. Ashton Yates, Esq.
Ludlow—T. A. Knight, Esq.
Maidenhead—R. Goolden, Esq., F.L.S.
Maidstone—
Clement T. Smyth, Esq.
John Case, Esq.
Malmesbury—B. C. Thomas, Esq.
Manchester Local Association—
G. W. Wood, Esq., *Chairman*.
Benj. Heywood, Esq., *Treas.*
T. W. Winstanley, Esq., *Hon. Sec.*
Sir G. Philips, Bart., M.P.
Benj. Gott, Esq.
Merthyr Tydvil—J. J. Guest, Esq., M.P.
Minchinhampton—John Ball, Esq.
Monmouth—J. H. Moggridge, Esq.
Neath—John Rowland, Esq.
Newcastle—Rev. W. Turner.
Newport, Isle of Wight—
Abr. Clarke, Esq.
T. Cooke, Jun., Esq.
R. G. Kirkpatrick, Esq.
Newport Pagnell—J. Miller, Esq.
Newtown, Montgomeryshire—
William Pugh, Esq.
Norwich—Right Hon. Lord Suffield.
Rich. Bacon, Esq.
Oxford—Dr. Daubeny, F.R.S., Professor of Chemistry.
Rev. Professor Powell
Rev. John Jordan, B.A.
Rev. R. Walker, M.A., F.R.S.
E. W. Head, Esq., M.A.
W. R. Browne, Esq., M.A.
Penang—Sir B. H. Malkin.
Plymouth—H. Woollcombe, Esq., F.A.S., *Chairman*.
Snow Harris, Esq., F.R.S.
E. Moore, M.D., F.L.S., *Sec.*
G. Wightwick, Esq.
Presteign—Dr. A. W. Davies, M.D.
Ripon—Rev. H. P. Hamilton, A.M., F.R.S. and G.S.
Rev. P. Ewart, M.A.
Ruthin—Rev. the Warden of.
Humphreys Jones, Esq.
Ryde, Isle of Wight—
Sir Rd. Simeon, Bart., M.P.
Sheffield—J. H. Abraham, Esq.
Shepton Mallet—
G. F. Burroughs, Esq.
Shrewsbury—R. A. Slaney, Esq., M.P.
South Petherton—
J. Nicholetts, Esq.
St. Asaph—Rev. Geo. Strong.
Stockport—Henry Marsland, Esq., *Treasurer*.
Henry Coppock, Esq., *Sec.*
Tavistock—Rev. W. Evans.
John Rundle, Esq.
Truro—Richard Taunton, M.D.
Henry Sewell Stokes, Esq.
Tunbridge Wells—Dr. Yeats, M.D.
Warwick—Dr. Conolly.
The Rev. W. Field, (*Leam.*)
Waterford—Sir John Newport, Bart.
Wolverhampton—J. Pearson, Esq.
Worcester—Dr. Corbet, M.D.
Dr. Hastings, M.D.
C. H. Hebb, Esq.
Wrexham—Thomas Edgworth, Esq.
J. E. Bowman, Esq., F.L.S., *Treasurer*.
Major William Lloyd.
Yarmouth—C. E. Rumbold, Esq., M.P.
Dawson Turner, Esq.
York—Rev. J. Kenrick, A.M.



Published by Baldwin & Cradock Paternoster Row London. May 1834.

BIBLIOTECA NAZIONALE VITTORIO EMANUELE III NAPOLI



Published by Baldwin & Cradock Paternoster Row London May 1834.

NAPOLI

Pl. 3

Fig. 1.

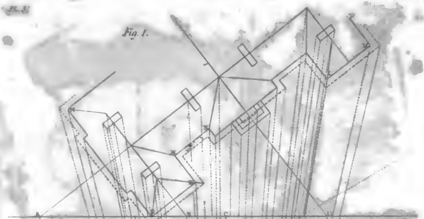


Fig. 2.

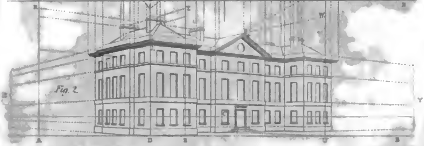


Fig. 3.

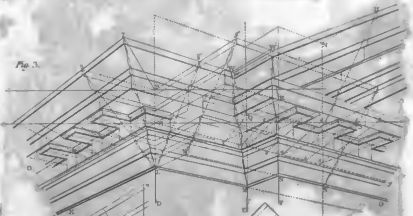


Fig. 4.

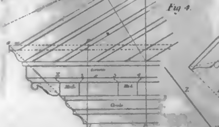
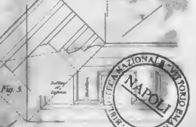
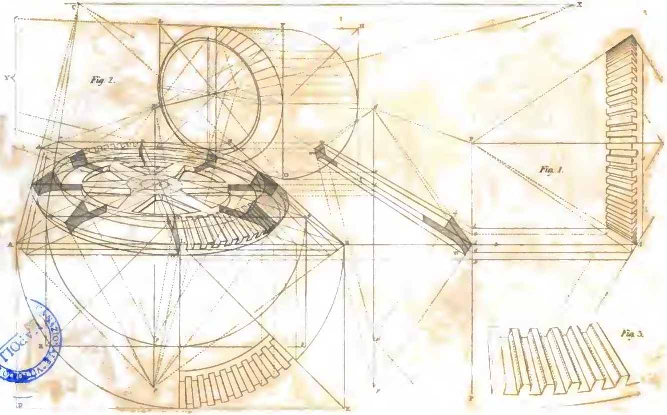


Fig. 5.



Published by Baldwin & Cradock, Paternoster Row London, May 1834.

BIBLIOTECA NAZIONALE VITTORIO EMANUELE III
NAPOLI



Published by Baldwin & Cradock, Paternoster Row London. May 1834

NAZIONALE VITTORIO EMANUELE III NAPOLI

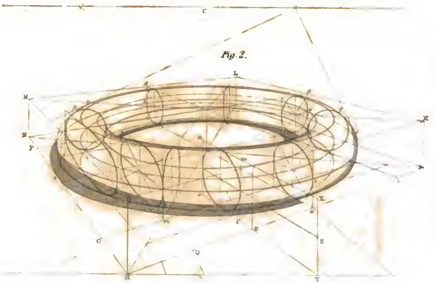


Fig. 2.

Fig. 5.

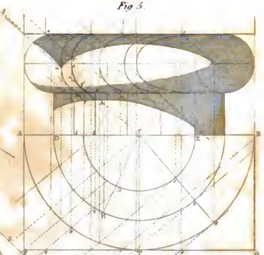


Fig. 1.

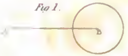
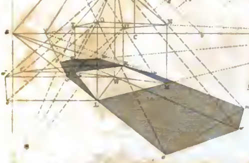


Fig. 3.



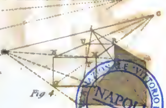
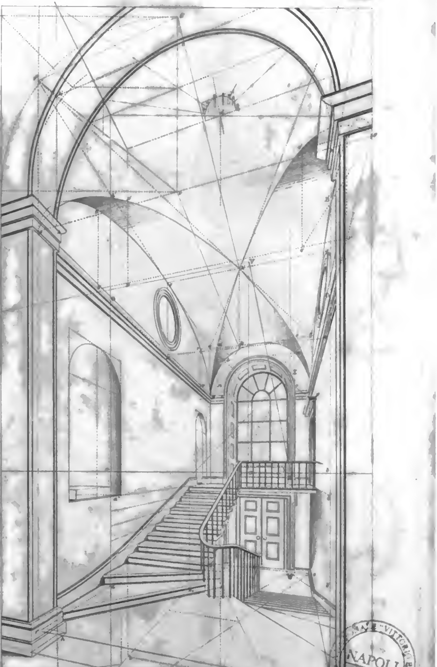


Fig. 4.

Published by Baldwin & Cradock, Paternoster Row, London, May 1834.

BIBLIOTECA NAZIONALE VITTORIO EMANUELE III NAPOLI



Published by Baldwin & Cradock, Paternoster Row London, May 1830.

BIBLIOTECA NAZIONALE VITTORIO EMANUELE III NAPOLI

Fig. 3.

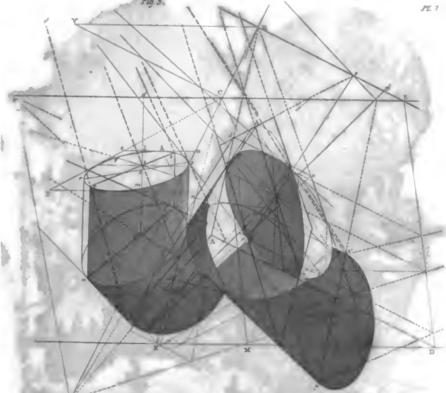
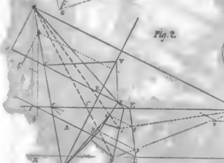


Fig. 4.

Fig. 2.



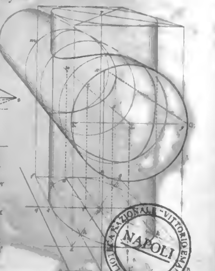
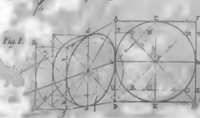


Fig. 1.



Published by Baldwin & Cradock, Paternoster Row London. May 1834

BIBLIOTECA NAZIONALE VITTORIO EMANUELE III NAPOLI

Fig. 1.

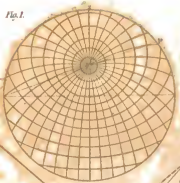


Fig. 2.

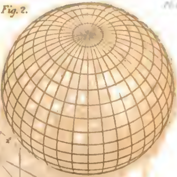
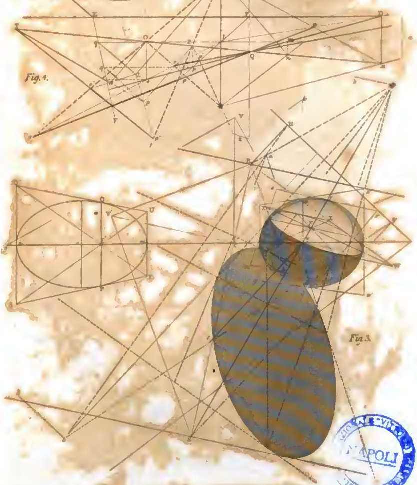


Fig. 4.

Fig. 3.



Published by Baldwin & Cradock, Paternoster Row, London, May 1834.

NAPOLI

PRACTICAL GEOMETRY.

INTRODUCTION.

THE object of theoretical geometry being the relations of extension or magnitude abstracted from all considerations of *matter,* the theorems of that science may be investigated without the aid of figures or diagrams, or at least such figures as suggest the idea of the relative magnitudes under consideration are sufficient; but when the principles thus deduced are applied in the arts, it is generally for the purpose of defining, or representing the form and magnitude of *matter* of some kind, so as to enable us to reason on the means of adapting it to the purposes for which it is intended: it is, therefore, necessary that this representation of the form should be correct. This specific application is called *practical geometry.*

In some of the mechanic arts, the outline or figure into which the material is to be formed is traced or drawn upon it, as is done by the carpenter and cabinet-maker; but on most occasions a kind of picture or representation of the figure, commonly termed a *working drawing,* is given to the mechanic as a guide: of this description are the plans and elevations furnished by the architect to the mason or bricklayer, which instruct them how to raise the walls of a building, or the carpenter how to arrange the timber-work of it. Such, likewise, are the drawings by means of which the various parts of a machine or instrument are modelled and cast by the fŏunder, and subsequently put together by the mill-wright or engineer.

But in the application of that branch of drawing called *perspective,* more complicated constructions than are ever required on the occasions before enumerated are constantly necessary: to the artist, therefore, a knowledge of practical geometry is especially essential.

The great difference between theoretical and practical geometry arises from the two following considerations:—

The *problems* of the theoretical science are only propositions put in a synthetical form, on account of the peculiar method of progressive reasoning employed; hence no principle or theorem can be applied to the solution of a problem which has not been previously demonstrated; but when these problems are applied in the arts, every principle may be had recourse to, that will facilitate the object in view, without regard to that order of arrangement which was necessary in the mathematical investigation.

Hence several rules for the solution of the same *practical* problem can be given, applicable on different occasions, according to the particular conditions of the case, and by the judicious employment of which accuracy of result may be easily and expeditiously attained; while many *theoretical* problems are solved, from the reason above mentioned, by methods which would never, under any circumstances, be made use of in practice.

Thus, for example, the mode of dividing a right line into any number of equal parts, as given in the theoretical problem, is far too circuitous and inaccurate in practice ever to be made use of, (Geo. I. § 7, Pr. 49:) and as it is a problem of perpetual occurrence, and of great importance, not only in geometrical drawings, but in other arts, some method of solving it, at once correct in principle and easy in application, must obviously be given in a work on practical geometry.

The various instruments contrived for the purpose of saving time and trouble in the application of geometrical theorems, concur to modify the solutions of problems still further. By means of these instruments not only is time saved, but much more accuracy is attainable than could be accomplished without them by the most careful application of theoretical principles; and since the measurement or delineation of absolute magnitude or extension is the object in view, accuracy, in a greater or less degree, is the first and most important consideration in all geometrical constructions: consequently those solutions should only be given in a work on this subject which admit of accuracy of result, while in theoretical solutions no such restriction is necessary, these being considered as perfect, if their mathematical truth can be demonstrated.

This treatise is intended to contain a collection of such practical problems as are most useful, not only to the draughtsman, but to those employed in mechanic arts; and, consequently, they will be applied on very different surfaces, and with very different instrnments. The carpenter constructs the figure he wants on the board or plank he is going to employ; he makes use of the leg of his compasses to draw lines against the edge of his square or pocket rule, and drives a nail to mark a point: the bricklayer and military engineer *set out* the plan of a house or of a *field* work, by means of strings stretched from pegs stuck in the ground; but artists, architects, or engineers, are the persons by whom practical geometry is principally employed in making carefully finished and elaborate drawings on paper, and who employ for the purpose what are hence called mathematical instruments.

It is chiefly for these last-named persons, and with reference to these materials and instruments, that the instructions in this work are written; but the principles being the same, the modification in their application, admissible when a less degree of nicety is required, will be easily understood.

It having been presumed that many persons may be desirous of possessing instructions for the solution of those geometrical problems which they may have occasion to apply, without being interested in the theoretical principles on which they are founded, the rules are given without any demonstration, yet so as to be understood by those not much conversant with mathematics. But for the sake of those who are unwilling to apply a rule without being acquainted with the reason of it, reference for the demonstrations of the theorems on which the rules are founded, is given, either to the Treatise on Geometry of the "Library of Useful Knowledge," or to some other elementary work attainable by most readers, and found in the libraries of all "mechanics' institutions," or similar establishments; and in those cases where such reference could not be made, or where some curious and useful observations on the problems suggested themselves, a note has been added, in which these or the demonstration will be found.

EXPLANATION OF THE REFERENCES.

(Geo. I. § 7, Prop. 58.) Geometry, Plane, Solid, and Spherical. Library of Useful Knowledge.

(Leslie, Anal. I. Prop. 7.) Elements of Geometry and Geometrical Analysis. By John Leslie, &c. Second edition.

(Leslie, G. C. L. Pr. 1.) Geometry of Curve Lines. By the same.

(Euc. I. Pr. 45.) Euclid's Elements. By Robert Simson.

OF THE INSTRUMENTS AND MATERIALS EMPLOYED IN PRACTICAL GEOMETRY.

The great modification in the practical construction of geometrical problems, caused, as has been mentioned, by the use of many instruments, renders it necessary to give an account and description of these before proceeding to the problems themselves. A set of these, containing what are required in geometrical constructions generally, is familiarly known by the name of a *case of mathematical instruments*. Together with such a case, and some others which it cannot contain, which will be presently described, the draughtsman should be provided with two or three drawing-boards of different sizes, made very flat and smooth on the face, and having their edges accurately at right angles: these boards are best when made of mahogany or oak, deal being more liable to warp or *cast*, and, from being soft, the points of the compasses in time injure the surface, which renders it impossible to draw on such boards neatly or accurately.

To ascertain whether the edges of a drawing-board are correctly at right angles, lines should be drawn by a T square applied to two adjoining sides, care being taken that the same edge of the square is used, and that the same side is uppermost: the right angles formed by these lines must then be examined by any of the constructions given in Problem 1, and if they are found to be correct, the board is a true *rectangle*, or is *square*, as it is termed. The lines for this purpose should be drawn in different parts of the board, some very near the edge, others in the middle. The principle of this mode of verification will be found explained in the description of the T square.

Of instruments, *compasses* or *dividers* claim the first notice, not only as being the most indispensable, but also because accuracy chiefly depends on the right use of them. As, however, these instruments are too well known to require particular description, some general observations on them are alone necessary.

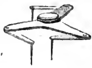
It may be regarded as a constant rule, that whenever any part of a construction can be made by the compasses, recourse should be had to them in preference to a *rule* or *straight edge*, especially when accuracy is required. A point determined by the intersection of two arcs approaches as nearly as possible to theoretical precision; and if the draughtsman be at all careful, a series of such intersections, mutually depending on each other, may be determined, the result of which, when the operation is verified by some other test, shall not err by any quantity visible to the naked eye. For this reason, solutions to several problems will be given in this treatise, for which these instruments are alone required, where such a construction is not too elaborate to be useful. The points of all compasses should be kept very sharp; if handled lightly, they will not pierce the paper, which they should not do if it can be avoided, because the dot thus formed is too indefinite; but if the points be not sharp, they would slip on the surface of the paper or board, whenever the legs were opened so wide as to cause them to rest very obliquely on it, and by this the distance intended to be set off would be altered.

Habit alone will enable the beginner to acquire the right manner of holding and managing the compasses. The only general directions that can be given are, that the *head* or joint should be held between the thumb and two first fingers of one hand; the thumb and fore-finger are sufficient to turn the instrument round, as is required in *stepping* a distance along a line when dividing it into equal parts. The legs should never be touched after the compasses are set, otherwise the opening might be altered by the pressure of the hand; and they should be held upright when applied to the paper, to obviate the slipping of the points as much as possible.

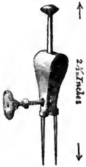
Compasses are generally made with one leg removeable, that a *steel pen*, as it is termed, may be inserted in its place, in order to describe circles, or a contrivance for carrying a black lead pencil for a similar purpose.

Hair dividers, as they are called, are compasses, one steel leg of which is formed with a spring, screwed to the upper part of the brass socket near the joint, and lies in a groove to admit of the legs being closed; the point is kept steady by a small finger-screw, which passes through the brass socket into the steel leg near where this is inserted in common dividers; this screw counteracts the spring. When a distance is taken nearly in the usual way in these instruments, the point can be accurately adjusted by the screw to the true distance, without the necessity of altering the angle at the joint, which cannot be easily done for any small quantity. The common spring dividers, however, are simpler and surer.

Triangle compasses are made with three legs, by means of which a triangle can be copied at once, a point being set to each angle. An improved form of these is shown in the figure, and the mode of using it will be readily understood without description.



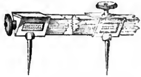
Besides the common kind found in cases of instruments, it is desirable to have two or three pair of *spring dividers*, which are adjusted by a screw, and can be retained for any length of time at precisely the same opening. There are few geometrical constructions in which some distances are not required more than once, and much time is saved, and inaccuracy avoided, by preserving these distances unaltered, instead of having to set the compasses to them every time they are wanted, which must be done if the draughtsman has not more than one pair, and cannot be done accurately by common compasses at all.



These dividers should be of different sizes: one pair, of which the annexed figure is recommended as a good form, being intended for minute divisions, not exceeding one-fourth of an inch; the others may be like those in common use, only better made.

Compasses to carry in the pocket are made in many forms, the legs being furnished with a steel pen, and contrivance for holding a pencil at the other ends, which can be substituted for the points when required; the legs are either put into a sheath, or double on a joint, so as to lie alongside of the socket, in order to avoid accidents. These are termed *pillar compasses*.

Beam compasses are for the purpose of setting off or transferring greater distances than can be done by the common compasses.



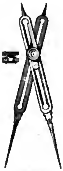
This instrument consists of a bar of wood, or metal, one end of which carries a point, admitting of slow motion by an adjusting screw; the other point is carried on a socket which slides, and can be clamped by a screw at any part of the bar when the required distance has been taken, nearly, between the points: the correction is made by the adjusting screw. The bar should be from thirty to forty inches long, with a scale divided on one side; the socket of the end point should have a vernier on the bevelled edge of an aperture in one side to apply to the divisions, and that of the other should have a mark in a similar bevelled edge to set to a division.

The points must admit of an adjustment in their sockets, by turning on their axes with a small eccentric motion, so that the distance comprised between them may be correctly that indicated on the scale by the zero marks of the bevelled edges. To ascertain this, take any distance in the compasses, and then compare it with another scale, known to be correct: if the points contain a different distance on this scale from that shown on the beam, they must be altered till the two agree.

To explain the mode of using the scale on the beam, suppose it were required to set off a length of 17·74 inches upon a line: the sliding socket must be moved along the bar till the zero mark is opposite the seventh division beyond 17 inches; it must then be clamped by the screw; the zero mark of the vernier* must then be brought, by means of the adjusting screw, till the *fourth* division coincides with that of the scale in the contrary direction, or backwards, from that marked 0: by this means a quantity equal to ·04 of an inch will be added to the 17·70 inches first set, and the whole distance required will be comprised between the points, provided these were previously verified, as was above explained.

In the figure the bar is supposed to be of wood, and the whole instrument is of the ordinary construction; but for nice operations, these compasses are made with metal beams, and the scale is very beautifully divided, both sockets carrying screws for slow motion, with magnifying glasses to examine the coincidences of the divisions, so that the utmost precision may be obtained. It is essential in this case that the beam should be quite straight, or the scale will not indicate the true distance between the points: but when beam compasses are only required to transfer long distances, it is immaterial whether the beam be straight or not.

Proportional compasses are intended to facilitate the construction of *similar* figures, (Geo. II. § 4,) a problem of constant occurrence. The form of this instrument will be best understood from the annexed figure: it is a species of double compasses, consisting of two equal parts, terminating in points at both ends; these open on a moveable centre, or joint, by which they are kept together, and which slides in an open groove: this centre can be clamped by the screw at any distance from the points. It will be easily perceived that the distances included between the points at the opposite ends of the instrument will be more or less unequal in proportion as this centre of motion is farther or nearer the middle of the length of the compasses. (Note A.)



There are divided scales along the sides of the groove for setting the centre to the proper place at once, so as to give the required proportion between the opposite distances; but these scales being in time rendered inaccurate from the wearing of the points shortening the length of the legs, they must only be employed to set the centre by nearly, and the correction must be made by trial. When so adjusted, as long as the centre is kept unaltered, all distances taken between the points at one end will be in the same constant proportion to those contained between the opposite pair.

---

* The principle and application of the vernier will be found briefly explained in Note C.

It must be remarked, that the joints of all compasses should not be screwed too tight, for this renders it difficult to set the instruments to a given opening with precision, owing to the exertion necessary to overcome the friction; and unless the joint is as much too slack, there is no danger of any distance being changed by holding the compasses, if they are handled properly.

Rules, or straight edges, for drawing right lines by, are made of ebony, box, mahogany, ivory, brass, or steel; and of any length, to the pleasure of the person requiring them, or according to the purpose for which they are wanted. Those of wood or ivory are lighter and neater for drawings on paper, while the metal ones, as less liable to *cast*, are more accurate.

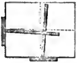
As a rule is useless if the edge be not perfectly straight, it should be proved occasionally by the following method:—A line must be carefully drawn on a very flat surface, with a sharp point, by the edge to be verified; the rule must then be reversed, so that the same edge may be again applied to the line, but on the other side of it, as shown in the figure, 1 2 being the first, and 2 1 the second position, the extremities of the edge being carefully set to coincide with those of the line, A new line should then be drawn as before, and if this be found identical with the first, the rule is *true;* but if they do not coincide in the middle as well as at the ends, a space is enclosed, and they cannot be right lines: consequently the rule is not to be trusted to.

A slight defect in a rule may be removed by drawing a fine file several times gently and very carefully along the faulty edge, and the rule must be subsequently verified.

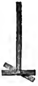
The T *square* consists of a thin rule, called the *blade*, fixed at right angles into a shorter and thicker piece, called the *stock*, so as to resemble in form the letter whence it derives its name. If the stock be applied against the side of a rectangular drawing-board, so that the blade lie flat on its surface, the additional thickness of the stock admits of its being moved along the edge of the board, and the blade is thus moved always parallel to its former positions: parallel lines at any distances may, by this means, be drawn by its edge. If the square be then applied to another side perpendicular to the first, the blade will obviously lie at right angles to its former position; and parallels, perpendicular to the former, may thus be drawn by means of the same edge as before.

This instrument is indispensable to architectural draughtsmen, in drawing plans or elevations of buildings, where the lines are numerous, and all parallel among themselves. The stock of the square being guided by the left hand, and a pencil in the right, will trace the lines of a cornice in one-tenth of the time which would be required to draw them by any other means; but in using the square, especially if it is a long one, care must be taken to hold the stock firmly against the edge of the board, as a slight pressure by the pencil at the end of the blade will otherwise force it out of the straight line. It is often, for this reason, necessary to hold the blade down by the left hand, in order to avoid this source of error. This precaution must especially be taken when many lines are required to be drawn very near the edge of the board, because, in such a position, half the stock only rests against the side, and consequently offers less resistance to the pressure of the pencil on the edge of the blade.

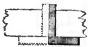
It has been already observed, that the edges of a drawing-board should be very smooth and true, and correctly at right angles: if this be the case, and the square be always applied to the same two sides, and the same edge made use of during the progress of a drawing, it is not necessary that the blade and stock should be at right angles, provided that the former be firmly fixed in the latter without any shake. To explain this, let the dotted lines in the figure represent the true perpendiculars to the edges of the board, or those with which the blade would coincide if it were correctly at right angles to the stock. The angle of error of the square, that by which its edge differs from the true perpendicular, is constant in both positions of the instrument, and lines drawn by means of it will be correctly vertical to each other, though not perpendicular to the edges of the board, provided these are truly at right angles; this will be readily perceived from the diagram.



The stock of a T square is sometimes made of two equal pieces, kept together by a finger-screw; the blade is fixed into one of these, *flush* with its inner face. If the other be applied to the edge of the drawing-board, the former, with the blade, can be turned on the screw, as a centre, to any angle; the screw being then tightened, parallels forming that angle with the side of the board can be drawn; and if applied to an adjoining side, the blade will lie at a right angle to its first position, on the principle just explained. Such a square is called a *bevel*.



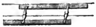
The square used by carpenters is formed like an L, or a T



square with half a stock. For the purposes to which this is applied, it is requisite that the instrument should be a true right angle; this may be ascertained by drawing a line by the edge of the blade, and then reversing it, as shown by dotted lines in the figure. If the line and blade in this new position coincide, it is correctly perpendicular to the edge of the board, and the square is *true*.

A rule made in the form of a right-angled triangle is of use for drawing perpendiculars, by applying one side to the given line, and the perpendicular required is to be drawn by the edge of that at right angles to the first. In Problem 2, a mode of drawing parallel lines by such a rule is described. It may be proved in the same manner as the carpenter's square. A small triangular rule of ivory is employed by architects, &c. for drawing very short perpendiculars, to the edge of the T square, against which it is applied.

There are two kinds of *parallel rules*, or instruments for drawing parallel lines: the first consists of a single rule, with an axis carrying two small rollers, fixed to one side; these must be of precisely equal diameters, and should be as far apart as the length of the rule will allow: the instrument, by rolling on these, will obviously be carried parallel to any position it was first placed in, and, consequently, parallels to any line to which its edge is set may be drawn by means of it. The edges of the rollers are toothed, or notched, to prevent them from slipping instead of rolling on the paper, for if they did the parallelism would be destroyed.



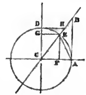
The second kind of parallel rule, shown in the figure, consists of two plain rules, connected by two equal pieces of brass turning on centres, and these must be truly parallel, so that in every position the four centres of motion may form a parallelogram. When used, the edge of one being set to any line, the other rule must be firmly held down by the hand, and the first moved till the same edge is brought to where the required parallel to the given line is to be drawn: this second rule must then be held by the hand till the other is at liberty to use the pencil. As it is obvious that if either rule slip during this operation the parallelism will be lost, these instruments should only be used for short lines, or where slight errors will not be of much importance.

In using rules, a little practice is required to enable the beginner to judge how far from two points, through which the line is to pass, the instrument must be set, to allow for the thickness of the pen or pencil; and also for keeping these last at the same angle, so that the line drawn by them may touch the edge, or be equally far from it, in every part, and consequently straight.

The faces of all rules may have divided lines, or *scales*, on them, of the various kinds required in geometry or drawing. In Problem 4, and in the subjoined note, an explanation will be found of the different modes of division of right lines into equal parts; but besides these, other divided lines, connected with angles and arcs of circles, constantly required in geometrical constructions, and found on most rules or *scales*, as they are hence called, must now be described.

The measure of the inclination of two right lines which meet in a point, or of the *angle* formed by them, (Geo. I. Def. 9,) is the arc of a circle of any radius, described from that point as a centre, and intercepted by the two lines. It is demonstrated (Geo. III., Def. 13, Prop. 33, Cor. 1) that the arc thus intercepted bears the same proportion to the whole circumference, whatever may be the radius of it, and consequently any circle may be made use of to measure an angle.

The circumferences of all such circles are supposed to be divided into 360 equal parts, called *degrees*, each of which is subdivided into 60 parts, called *minutes*, and each of these again into 60 parts, called *seconds*. The measure of an angle is expressed by the number of degrees, minutes, and seconds contained in the intercepted arc: the marks $^{\circ}\ '\ ''$ are used to distinguish these divisions. Thus $75^{\circ}\ 16'\ 24''$ means an arc, or angle, of seventy-five degrees, sixteen minutes, and twenty-four seconds.



Let C be the angular point in which the two lines A C, B C meet; if a circle *of any* radius be described from C, the circumference being supposed divided into degrees, &c., then the number of these contained in the arc A E is the measure of the angle A C B, and obviously shows the proportion of that arc to the whole circumference.

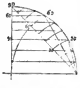
Produce A C till it form a diameter to the circle, and draw C D perpendicular to it; then A C D, or a right angle, being measured by one-fourth of the circumference, will contain 90°. From the point E draw E F, E G perpendicular to A C, D C, and draw A B also at right angles to A C, or parallel to D C, F E; join A E by the *chord*. (Geo. III. Def. I.)

Then A C is the radius
A E — chord
F E — sine
A F — versed sine
A B — tangent
C B — secant
of the angle A C B, or arc A E.

The angle D C B, or the difference between the given angle and a right angle, is called the *complement* of the given angle; and the sine E G, the tangent D H, and the secant C H of that complement are called the co-sine, co-tangent, and co-secant of the original angle A C B.

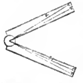
These lines, drawn with reference to the arc of any other circle, described from C as a centre, intercepted by C A, C B, will, by the properties of similar triangles, be in the same proportion to the radius of that circle that those just enumerated bear to C A; if, therefore, any such radius be taken as the unit, the length of these various lines may be expressed in parts of it. This is what is done in the divided lines on common scales, by means of which angles of any given number of degrees and minutes may be constructed.

The line most commonly made use of for this purpose, for reasons explained in Prob. 3, is that called the *line of chords*, which contains the length of the chord of every degree of the *quadrant*, or one-fourth of the circumference transferred to a right line, in the manner shown in the annexed figure. As the chord of an arc of 60°, or of one-sixth of the circumference, is equal to the radius, (Geo. III., Prop. 28.) in order to construct an angle of any given number of degrees, the length of the chord of 60° must be taken in the compasses from a line of chords, and an arc of a circle described with it as a radius from the point where the angle is required; the chord of the given degrees and minutes must then be taken from the same scale, and must be set off on the arc; and lines drawn from the centre through the points thus marked will form, or contain, the required angle.



The tangent of 45° being also equal to the radius of the circle, that length may be taken from a *line of tangents*, and being set along a given line from the angular point, a perpendicular must be drawn from the other extremity of the distance, which being made in length equal to the tangent of the given number of degrees and minutes taken from the same scale, a line drawn from the extremity of this perpendicular through the centre will form the required angle. But this construction is longer and less accurate than that by the line of chords; and is, therefore, only employed on particular occasions, as when the right angle is already drawn for some other purpose.

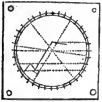
All the divided lines on common rules are necessarily unalterable, and can only therefore be made use of when they happen to be adapted to the degree of magnitude, or scale, on which the construction is to be made. The instrument called a *sector* is intended to obviate this deficiency of common scales: it consists of a jointed rule, like a carpenter's common two-foot rule, which it exactly resembles, but is better made; on each leg corresponding lines are drawn from the centre of the hinge, and each pair of lines is similarly divided, either into equal parts, or into the lines of chords, tangents, secants, &c. above described. To explain the use of this instrument, suppose a line of chords was wanted to set off some angles on a circle of a given radius, larger than, or different from those by which the lines of chords on any scales in the draughtsman's possession were determined; the given radius being taken in the compasses, the sector must be opened till the divisions marked 60° on the corresponding lines of chords on its legs are at that distance apart, or so that the points of the compasses accurately fall into those marks. When thus adjusted, the angle of the instrument must be kept carefully unaltered; the required degrees of which the angles are to consist are then to be taken from the similarly numbered divisions on the corresponding lines of the two legs of the instrument: thus, if the angle required were $23\frac{1}{2}$°, one point must be set in the division $23\frac{1}{2}$ on one line, and the legs of the compasses opened till the other point falls into the $23\frac{1}{2}$ division of the other leg of the sector; and so of any given number of degrees. In the same way, the lines of tangents, &c. are to be used. The principle on which this instrument is employed is demonstrated in Geo. II. Prop. 29, 31; the parallel chords, taken across the angle made by the legs of the sector, being proportionate to the corresponding radii, or the lengths of the divided lines from the centre of the hinge, as has been also before mentioned.



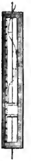
The *protractor* is another instrument for constructing angles of a given magnitude. This is made in two forms: a mark in the centre of one edge of a plain rectangular scale is supposed to be the centre of a semicircle; the degrees, &c. into which this is divided being transferred to the three other edges of the scale by lines drawn to this centre from the circumference. To use this instrument, the first or plain edge is applied to a given line, intended for one leg of the angle, so that the central mark shall coincide with the angular point; a dot being then marked at the given number of degrees, as found on one of the other edges, a line drawn through this mark and the first point will obviously make, with the given line, the angle required. This form of protractor is of great use for drawing short perpendiculars; the central mark and the 90° division being set so as to be on a given line, the edges of the scale will be at right angles to it, and the perpendicular may thus be drawn.

The shortness of the radius of the semicircle represented by the edges of this protractor, and the mode of its construction, as well as that of using it, being sources of error, it should not be employed in nice operations. The *circular protractor* is therefore to be preferred. This is a flat thin circle of brass, with its outer edge divided into degrees, the bevelled edge of a bar across it being a true diameter to this graduated circle, and having a mark at its centre; this mark and edge being set to the angular point in a given line, the required number of degrees is marked off from the rim as before. The circular protractor is sometimes made with a moveable radius, which can be set to the required number of degrees on the edge, and the second line to form the angle drawn at once by its edge, without removing the instrument. For greater precision, the radius is provided with a vernier, to subdivide the primary divisions.

A very ingenious protractor is made use of in plotting the government surveys, executed under the direction of the Board of Ordnance, which we are allowed to describe. A circle, about a foot in diameter, is engraved on a copper plate, and very accurately divided into degrees, and numbered; impressions are printed from this on thin card paper, and the circle cut out from them, the part used being the square sheet with the circular aperture; for this reason the divisions are engraved outwards from the circle on the plate. The angles taken in surveying with a compass are all measured from a fixed line, namely, the magnetic meridian; consequently, in laying down, or, as it is termed, in plotting such a survey, all the angles are to be set off from parallel lines representing that meridian. It is to facilitate this proceeding that the printed impressions are employed in the following manner:—One of them is fixed down flat on a drawing-board, by *drawing-pins*, and so that the diameter, which would join the zero and the 180° division, may represent the meridian. In order to draw lines through any given points which shall make the necessary angles with this meridian, the edge of a parallel rule is laid across the circle to the given number of degrees, taken in opposite directions from it, so as to lie in a second diameter, the edge of the rule being then moved in the usual manner to the point, a line is drawn which will obviously form the required angle with the first diameter, (Geo. I. Pr. 18.) In this manner the angles are formed at each successive point that falls within the aperture; the card is then moved to a new position, care being taken that the meridional diameter shall be truly parallel to its former position; the operations are then continued, and when the impression is worn out by use, another is substituted.



Another instrument for facilitating the plotting of surveys, the invention of Sergeant Doul of the Engineers, is also used under the same authority. In laying down a plan or map from a survey, the lengths of perpendiculars at different points in a given line are perpetually required to be set off. Commonly these distances are marked along the first line from a scale, the different perpendiculars are then drawn, and the lengths of these taken in succession from the same scale; the instrument in question is intended to shorten this operation. It consists, as may be seen in the figure, of a rectangular brass frame, within which a square piece slides freely; one of the longer edges of the aperture is divided into inches and tenths, and the sliding piece carries a vernier on its edge, which applies to the divisions; the cross edges of this latter are bevelled, and are also graduated from the centre of each outwards. At the exterior ends of the frame are marks, which, if set to a given line, the centres of the edges just mentioned will also coincide with it. If, therefore, the piece is successively moved to the proper distances, by means of the scales, along the aperture, the lengths of the perpendiculars may be set off, by means of the divisions on the bevelled edges of the sliding piece, on either side of the given line as required, without the necessity of drawing them, and a line is afterwards drawn through these points by hand.



These are the principal instruments required for geometrical drawing: others, more elaborate, such as *pentagraphs*, *centrolineads*, and *elliptographs*, will be noticed under the problems for the solution of which they are intended. One objection to these arises from the difficulty of adjusting them so as to describe the lines or curves they are calculated for, in the precise situation they are wanted on the drawing; it frequently happening that the time thus consumed nearly counterbalances that saved by their application, and some injury is always caused to the drawing by the necessary trials. The expense of these instruments also precludes many learners from possessing them; but this need not cause them any regret, since the necessity for having recourse to problems to perform what the instruments are meant to effect, will make them acquainted with some theorems with which they might not otherwise have been conversant.

An instrument called a *drawing-pen* is required for drawing, in ink, lines intended to be permanent. It consists of two thin equal steel pieces, tapering to points, like the nib of a common pen, and placed side by side; the small opening to be left between them, according to the strength of the line intended to be drawn by it, being regulated by a finger-screw passing through both pieces; a small quantity of fluid *Indian ink* must be inserted by a camel-hair pencil between the points, and the pen used with its flat side against a rule. Care must be taken that no ink is left on the outside of the points, for if there be, it will run on the rule and soil the drawing. When done with, the pen should be wiped dry, otherwise the ink will corrode the steel.

Common ink may be used when the drawing is not to be shaded or coloured, but lines drawn with it *run* when washed over by water colours, and will give a hard cold look to the outline.

A steel pen is always found in sets of instruments, which can be substituted for the leg of a pair of compasses for the purpose of drawing circles in ink.

In all complicated geometrical constructions, it is essential to have the surface on which they are made very flat, or, as nearly as possible, a true plane. To effect this, the paper employed must be *strained*, which can be done in either of the following ways:—

1. The paper must be thoroughly and equally damped with *clean* water, applied by a sponge, or large flat camel-hair pencil, on one side only, care being taken not to rub or injure the surface in the process. Some thin common glue, or strong paste, is then to be very equally spread, in a band of about half an inch, round the four edges of the other or dry side; each edge, as the glue or paste is put on, being pressed down on the drawing-board, so that the paper may everywhere adhere to it. When the paper contracts in drying, as it is firmly held at the edges by the glue or paste, all creases or undulations disappear from the surface, and it forms a smooth plane. Practice and care can alone enable the beginner to accomplish this successfully: the only general instructions that can be given are, that the paper should not be drawn too tight on the board while it is damp, or else it either tears, or parts from the board in places when dry; more especially if the paste or glue has been unequally spread, or if any part where this is put on has been wetted. It is also necessary, in order to make the paper lie as flat as possible, that opposite sides should be glued or pasted in succession, and then the paper drawn equally, so that all large creases or swellings may be removed: the two other sides are then to be treated in the same way. The object in view being that the glue or paste should set before the paper is quite dry, so that the latter, by its contraction, may remove all creases, and lie quite flat on the board, and yet that the glue should yield sufficiently to that contraction to prevent the paper from tearing.

2. Some drawing-boards are made of a pannel, held in a frame by two thin bars of wood, which slip into mortised holes at the

inner edges, the face of the pannel, when in its place, lying even with that of the frame. The pannel being taken out, the paper is damped, as before described, and being laid very evenly over it, the frame is pressed down, and the bars of wood are fixed in their places, so as to confine the pannel to the frame; as the paper contracts in drying, it is held round the edges by being squeezed between the pannel and frame: this contrivance supplying the place of the glue or paste used in the former method, and the same precautions must be taken not to draw the paper by the frame too tight over the pannel, in fixing it.

This is the neater and easier mode of straining drawing-paper; but the size is limited by that of the pannel of the board, whereas paper of any dimensions may be strained by means of glue or paste, provided the board is large enough.

If there is to be no tinting or colouring on the drawing, those lines of the construction which are not required to be visible may be drawn with the point of the compasses, or any steel point; but the surface of the paper is so injured by this, that any colour laid on it will *run* in those lines, and spoil the evenness of the tint. It is better, therefore, to use a hard black lead pencil, cut to a fine point, when the outline is to be subsequently shaded, as in architectural drawings, or drawings of machines.

There are some general rules, applicable in all geometrical constructions, by observing which accuracy will be obtained and much time saved. In all the following Problems these rules are supposed to be attended to, though to prevent unnecessary repetition the specific occasion of their application is not noticed.

1. Arcs of circles, or right lines by which an important point is to be found, should never intersect each other very obliquely, or at an angle of less than 15 or 20 degrees; and if this cannot be avoided, some other proceeding should be had recourse to, to define the point more precisely.

2. When one arc of a circle is described, and a point in it is to be determined by the intersection of another arc, this latter need not be drawn at all, but only the point marked off on the first. As it is always desirable to avoid the drawing of unnecessary lines, the same observation applies to a point to be determined on one straight line by the intersection of another.

3. Whenever the compasses can be used in any part of a construction, or to construct the whole problem, they are to be preferred to the rule, unless the process is much more circuitous; or unless the first rule forbids.

4. A right line should never be obtained by the prolongation of a very short one, unless some point in that prolongation is first found by some other means; especially in any essential part of a problem.

5. The larger the scale on which any problem, or any part of one, is constructed, the less liable is the result to error; hence all angles should be set off on the largest circles which circumstances will admit of being described; and the largest radius should be taken to describe the arcs by which a point is to be found through which a right line is to be drawn; and the greater attention is to be paid to this rule in proportion as that step of the problem under consideration is conducive to the correctness of the final result.

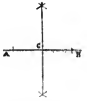
6. All lines, perpendicular or parallel to another, should be drawn long enough at once to obviate the necessity of producing them.

7. Whenever a line is required to be drawn *to* a point, in order to insure the coincidence of them, it is better to commence the line *from* the point; and if the line is to pass through two points, before drawing it, the pencil should be moved along the rule so as to ascertain whether the line will, when drawn, pass through them both; thus if several radii to a circle were required to pass through any number of points respectively, the lines should be begun from the centre of the circle; any error being more obvious when several lines meet in a point.

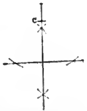
§ PROBLEMS RELATING TO RIGHT LINES AND ANGLES.

Problem I.

To draw a line perpendicular, or at right angles, to a given line A B, *through a given point* C.*



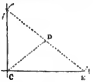
Case 1. If the point be in the given line: Set off from C each way along the line A B any convenient distance; from the points thus marked as centres, with any radius, describe arcs intersecting each other on both sides A B, if possible; then draw a line through the intersections and through C, which will be perpendicular to A B as required (Geo. 1. Prop. 44). This construction will be most accurate when the radius of the arcs is about equal to the distance between the centres.



Case 2. If the point be not in the line: With any length as a radius, as much greater than the distance of C from A B as is possible, describe arcs from C, cutting A B in two points, from which with any distance describe arcs on the other side of A B to C: a line drawn through C and the intersection of the arcs will be perpendicular to A B.

It will be better to describe arcs on both sides of the line, as in Case I.

Case 3. If the point be in the line A B, but near the end of it:



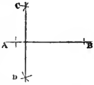
Assume a second point D at pleasure, any where out of A B; and from D, with D C for a radius, describe arcs, one cutting A B in E, and the other in the direction in which the perpendicular from C will lie: draw a line through E and D to cut the second arc, and a line drawn from the point of intersection through C will be at right angles to A B. (Geo. I. Prop. 44.)

---

* The size of these diagrams being limited by the form of this publication, the student need hardly be cautioned against taking them as a guide for the scale on which he ought to apply the problems in practice, this should be as large as circumstances will admit. It may also be mentioned here that the dotted lines indicate those which do not require being actually drawn, but are shown either to explain the proceeding, or for the purpose of demonstration in the notes.

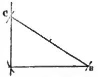
The point D should be taken, so that D C if drawn would form an angle of about 45° with A B, and as far from C as convenient: the line E D need not be drawn, it is sufficient if the point in which it would intersect the arc be marked, according to the second general rule.

Case 4. If the point be not in the line, but near the end of it:



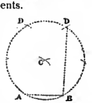
1. Mark any convenient point B in the given line, as far from C as possible; with B C as a radius, describe an arc on the other side of A B at D, and also mark on this line the point in which the arc would cut it: from this, as a centre, with the distance to C as a radius, intersect the arc described on the other side of A B in D; then a line drawn through C and this intersection D will be perpendicular to A B. (Geo. I. Prop. 45.)

2. Two points may be assumed in A B, one on each side of C if possible, then with the distance of C from each of these describe arcs intersecting on the other side of A B, and the perpendicular is to be drawn through the intersection and C. (Geo. III. Prop. 3, Cor. 3.)



3. Any point B being taken in the given line, join B C and bisect it; (see Prob. 4.) with the half of B C as a radius, describe an arc cutting A B, and draw the required perpendicular through C and the point of intersection. (Geo. I. Prop. 44.)

If the perpendicular required is to be a short one, or if no essential part of a subsequent construction depends on the accuracy of the right angle, a triangular rule, or a protractor, may be used, as was stated in the description of those instruments.



If A B be any two points given, a third D may be found, by means of the compasses alone; so that a line joining it and A or B would be perpendicular to a line passing through A and B.

From A and B as centres with any radius, A B for example, describe arcs cutting each other in C, and from C with the same radius describe a circle, which must necessarily pass through A and B; from either of these two points step the radius three times on the circumference to D, then a line joining D B or D A, according as A or B was used, would be perpendicular to A B. (See Note D.)

If, with any radius, arcs be described from each extremity on both sides of a given line A B, a line drawn through the intersections will not only be perpendicular to A B but will bisect it.

(Geo. I. Prop. 43, and Cor.) This construction is applicable whenever a circle is to be described that is to pass through the two points, as the centre must lie in the perpendicular bisecting the line joining them. (Geo. III. Prop. 3.)

To ascertain whether two given lines are at right angles to each other; from any point in one, taken as far as possible from their intersection, with any large radius describe arcs to cut the other line, and if these points of intersection are found to be exactly at equal distances on each side of that of the given lines, these are mutually perpendicular. — This is the converse of the construction for Case 1.

Builders make use of a method of setting off a line at right angles to a given one, which deserves notice here. Suppose the front of a house is indicated by a string stretched between two pegs stuck in the ground, and the side walls are required to be marked out by the same means, at right angles to the former; the corner of the building being marked by a peg. Three yards are carefully measured off along the string, and another peg fixed in the line: two other strings, one four yards and the other five yards long, are fixed, the former to the peg indicating the corner, and the other to that at the termination of the three yards; these strings being stretched tight, and their ends brought carefully together, the point where they meet will be in a line perpendicular to the first string. (Geo. I. § 7, Prob. 44, 3d method.) If the building be very large, and greater nicety is required, 6, 8, 10 yards, or any equal multiples of 3, 4, 5 may be used.

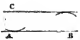
Draughtsmen sometimes make use of the same mode on paper, by taking 5, 4, 3, or any multiples of them, from any scale of equal parts; the larger number being always the hypotenuse of a right-angled triangle, of which the sides are of the length indicated by the two other numbers.

This problem being that of most frequent occurrence in practical geometry, the student should practise repeatedly all these constructions, so as to render himself familiar with their application, according to the particular circumstances of the case; and he will, by so doing, ascertain what parts of the operations may be abridged, and time thus saved.

Problem 2.

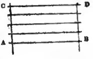
Through a given point C, to draw a right line parallel to a given right line A B.

1. Place one leg of the compasses in C, and open them till the other will describe an arc of a circle that will accurately touch

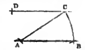


A B, without cutting it. From any other point B, taken at pleasure in the line, but as far as conveniently may be from the former, describe an arc with the same radius on that side of the line the given point is situated. A line drawn through C, and, as a tangent to or touching (Geo. III. Def. 7.) this second arc, will be parallel to the given line. (Geo. I. § 3, Prop. 16.)

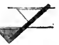
This is the only method to be depended on when the required parallel is more than two inches long, if accuracy is required. Care must be taken in obtaining the true perpendicular distance from the point to the line, and in preserving the opening of the compasses unchanged, to describe the arc from the second point with.



When several parallel lines at equal distances are required, it is best to draw two lines, A C, B D, parallel to each other, and as nearly at right angles to the given one A B, as may be estimated by the eye; the farther apart the better; the distance is to be *stept off* along A C, B D from A B, and the corresponding points joined by the parallels required.



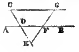
2. If the point C be far from A B, in proportion to the length of this, draw a line C A to the farthest extremity of A B, and from C, with C A as a radius, describe an arc; and from A with the same radius cut A B in a point B; from A, with B C for a radius, cut the first arc in D; and a line drawn through C and this intersection D will be parallel to A B. (Geo. I. Prop. 15, Cor. 2.)



3. If the parallels are short, place one side of a triangular rule to coincide with the given line; then to either of the other sides apply another plain rule, and holding this firmly down, so that it shall not slip, slide the triangular rule with the other hand, till the side of it, which was applied to the given line, touches the given point, and holding it steadily in that position, the required parallel may be drawn. Unless the rules are held so as not to slip or shift during the operation, and unless the triangle is slid very carefully along the side of the plain rule, the whole proceeding is useless; consequently, from the difficulty of fulfilling these conditions, this method should only be used for short lines, and in that case a parallel rule may also be employed.

Another mode of drawing a line through a given point, parallel to a given line, must be mentioned here; because, not requiring the description of any arcs, it may be applied in the field, where it is impracticable to draw a circle.

Let A B be the line and C the point. Through C draw a line to cut A B, at an angle of about 60° by estimation, and make D E, on the other side of A B, equal to C D; from E draw another line also to cut A B at nearly the same angle, and make F G equal to E F; then a line through C and G will be parallel to A B. (Geo. II. § 4, Prop. 29.)

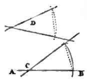


Problem 3.

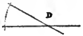
To construct an angle of a given magnitude.

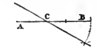
Case 1. At a given point C, in a given line A B, to make an angle equal to a given rectilinear angle D.

From the angular point of the given angle, with any, the largest convenient, radius describe an arc, cutting the legs, or mark off the length of that radius on them. From C, with precisely the same assumed radius, describe an arc from A B on that side on which the angle is to lie: take the chord of the arc on the given angle, or the distance between the points marked on its legs, and set it off from A B on the arc last described, and draw through C and the point thus found the right line, which will make with A B an angle equal to D. (Geo. I. § 7, Prop. 47.)



If the given angle D be greater than a right angle, it will be better to construct an angle at the given point C by the foregoing method, but on the contrary side of the line to that on which the angle is to lie, equal to the supplement (Geo. I. Note to Prop. 2.) of the given angle; which is found by producing one of the legs of the angle beyond the angular point; the leg of the angle thus constructed at the given point being produced will form the required angle in the situation where it is wanted. Or the supplemental angle may be constructed on the same side of the given line, but in the opposite direction, as will be easily understood from the figure.





Case 2. If the angle is to consist of a given number of degrees. One of the principal occasions on which this case is required is in

the laying down, or as it is termed the *plotting*, of the triangles of trigonometrical operations, in the construction of maps or plans, where many angles are to be drawn round one point. The best mode of proceeding is to make use of a table of natural sines, which is to be found in all volumes of logarithmic or nautical tables. A circle being described round the angular point as a centre, with the exact radius of ·5 taken from any convenient scale (say five inches from a scale of inches, ten being considered as unity); the natural sines of the halves of the given angles to be constructed are to be sought for in tbe table, and are to be taken from the same scale of equal parts and set off, in the proper succession, on the circumference of the circle; then radii drawn from its centre through the points thus found will form the angles required. (Note B.)

The advantage of this method is, that the compasses being alone required to set off the *chords*, the operation is simpler and more accurate than if the *sines* of the angles were made use of, which would require two processes; and when many angles are to be made at one point, and this has to be repeated often, the trouble of using the table is more than compensated for, by that saved in abridging the rest of the proceeding, and by the very superior accuracy of the construction.

The same principle is applicable to the converse of this problem, that is, to the measurement of how many degrees a given rectilinear angle consists. For if an arc be described from the angular point, with a radius of ·5, and the chord of the angle be taken from the same scale, tbe number thus found, sought in the table of natural sines, will give half the angle made by the lines with much greater accuracy, and very little more trouble, than if a scale of chords or a protractor were made use of.

On most other occasions the angle may be deduced from the division of the circumference of the circle, and that very readily, if the given number of degrees be some aliquot part of 360°, as is most commonly the case in geometrical or architectural drawing; even when it is not, and the required angle is some incommensurable part of the circumference, and is required to be laid off very accurately, the circle being described of a sufficiently large radius, the arc may be taken by estimation, between two contiguous divisions, more accurately than if the line of chords of common scales, or the sector were made use of.

If two right lines cross one another, the angle made by them may be measured by means of a *circular protractor*, without the necessity of setting the centre of the instrument to the angular point. For the number of degrees, intercepted by the lines forming the angles, being taken from the divided edge of the circle, half their sum, or half their difference, according as the angular point is within or without the circumference of the instrument, is the measure of the angle. (Geo. III. § 3, Prop. 19.)

A ready and accurate mode of dividing the circumference of a circle into any even number of parts, and into 5, is given in Problem 36.

PROBLEM 4.

To divide a given right line A B *into any required number of equal parts.*

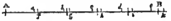
A distance taken in the compasses by estimation, as nearly equal to one of the required parts as possible, is to be *stepped* along the line very lightly from one end; if the distance was by chance taken correctly, the line will be divided by it at once; but, if not, the point of the compasses will not fall into the other extremity of the line, but will either exceed or fall short of the point, according as the estimated distance was taken greater or less than the true one.

Without raising the instrument from the paper, this error is to be divided by the eye into as many parts as the whole line is required to be divided into, and the compasses opened or shut the quantity of one of these smaller divisions, according as the first assumed distance was too small or too large.

The trial must be now repeated, and if the distance is still not correct, the compasses must be again altered by the estimated correction, as before, and so on till the error is very small; as these trials will be found, whatever care may have been taken, to have injured the surface of the paper by the number of dots made, which would cause confusion and error by the points of the compasses slipping into the former marks, the last distance should be stepped backwards from the other end of the line; the very minute error that may now exist had better be divided among the last divisions thus taken backwards by the eye, and this will ensure, practically, more accuracy than if further attempts were made to change the compasses by such a very small quantity.

As this problem is of such frequent occurrence, the neatness and correctness of the drawing will depend a good deal on attention to these instructions. To render them as clear as possible, the whole operation is shown in the diagram.

Suppose it were required to divide A B into five parts. The first trial is indicated by the dots *a*, *b*, *c*, *d*, *e*, and the distance taken by guess was too small, or was less than one-fifth of A B.



Now the leg of the compasses remaining in the dot d, the error e B is to be divided in the mind into five parts, and the compasses opened to include one of those divisions added to $d\ e$, or the assumed division, the leg in d not being moved.

The stepping being repeated from A, with this corrected distance, is indicated by the points $f\ g\ h\ i\ k$, and the correction was taken too large, or the whole opening of the compasses is more than one-fifth of A B; the compasses are to be shut therefore by one-fifth of B k, and a new trial made, which, if any care has been taken, will be exceedingly near the truth.

Now with this last corrected distance commence from B, and the new dots will fall *between* the former, and be therefore more distinguishable, and may be marked on the paper with more pressure of the hand on the compasses: if any error still exists it is to be divided, by eye, among the divisions, so as to increase or diminish each by an equal quantity, being one-fifth of the last error; and this final correction may be made with the point of a pencil.

When it is required to divide a long line into a considerable number of equal parts, it is best, if the number will admit of it, to resolve it into two factors, and first to divide the line into the number of equal parts indicated by the smaller factor, then subdivide each of these parts into the number expressed by the larger factor: thus, if it were required to divide a line into 15 equal parts, it is better to divide it first into 3, and then each of these into 5, since $3 \times 5 = 15$. If the given number be prime, such as 11, 13, 19, &c., of course the line can only be divided by trial.

It will be adviseable, when the number of divisions is large, and a prime number, to draw another line, making it accurately equal to the given one, to divide this by trial, till the correct part is got, and then to step this along the original.

If it be required to divide a line into two equal parts, or to bisect it, the compasses being opened to half the length of the line by guess, this distance should be set from each extremity, and the small error divided by eye, by the pencil point.[*]

In architectural and other mechanical drawings, it is perpetually requisite to set off on a line two different distances in alternate succession, as the distances from centre to centre of a series of columns or windows, and subsequently half the width, or the semidiameter of them, from the former points; or the width of the piers between the windows and that of the windows themselves. In these cases it is better to set off the distance, comprising both of the required ones in succession, and then, with the second distance in the compasses, to step from the points thus obtained the required spaces in the proper direction.

---

[*] These instructions may at first seem too trifling and superfluous; but when it is remembered that it is on attention to such minutiæ that the accuracy and neatness of geometrical drawings must chiefly depend, every circumstance by which those important objects can be attained, is deserving of attention.

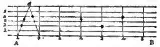
If a line is set out on the ground, by means of a string stretched from two pickets, it is very easily bisected, by disengaging the string at one end and doubling it, or by taking another equal in length and doubling that; the length of this doubled string set along the first will give the central point of it.

In the same manner it may be divided into three or more parts.

The most accurate mode of dividing a short line into equal parts is that used in the construction of scales, and is called diagonal division: it is an application of the mode of dividing a line given as the solution of the problem in Treatises on Geometry. (Geo. I. Prop. 49.)

If A B be a line divided into equal parts for a scale, it is always necessary that one of these divisions, at the end of the line, should be subdivided into as many parts as the unit of linear measure, represented by the scale, requires; thus if the divisions represent feet, one of them should be subdivided into 12 parts for inches.

Six lines, at equal distances at pleasure, must be drawn parallel to A B, and the divisions continued across them by lines, very truly parallel to each other, and perpendicular to A B; the last of these divisions, which is to be subdivided, is to be bisected on the farthest parallel, and two oblique lines drawn from each end of the division to the bisecting point, as 06, 16.

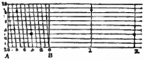


It will easily be seen, that these oblique lines, by their intersections with the parallels in succession, from A B to the farthest, and then back again, divide the space into 12 equal parts; (Geo. B. II. Props. 29. 31.) thus the small quantity from the perpendicular, 0, to the nearest oblique line on the 1st parallel, is $\frac{1}{12}$, on the 2nd parallel $\frac{2}{12}$, and so on to the 6th or farthest, which is by construction $\frac{1}{2}$ or $\frac{6}{12}$; then taken back in succession, the quantity from the perpendicular 0 to the farthest oblique line on the 5th parallel $\frac{7}{12}$, on the 4th parallel is $\frac{8}{12}$, and so on till the oblique line meets A B in 1 or $\frac{12}{12}$ of the whole space.

The quantity between the two dots on the 2nd parallel would be taken in the compasses for 3 feet 10 inches on a drawing made to this scale; the quantity between the dots on the 4th parallel represents 4 feet 4 inches, and those between the 3d and 4th parallel 2 feet 8½ inches;—the dots showing where the points of the compasses are to be placed to take these distances, if required.

The wider apart the parallels are drawn, by so much the more accurately can dimensions, such as the last, be measured off; the space between two adjacent parallels being subdivided by the eye.

This diagonal division is also employed in scales to subdivide into still minuter quantities, which could not be done by any other mode, with tolerable precision: this will easily he comprehended by means of the annexed figure.

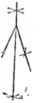


The unit A B is first subdivided into 10, and each of these again subdivided into 10 by the diagonals and parallels. If A B represented a scale of 100 feet to ½ an inch, each direct division of A B is 10 feet; and by means of the diagonals, distances can be taken to single feet, and by the eye to half a foot.

Thus the dots between the 8th and 9th parallel represent 178½ feet, those on the 3d parallel represent 243 feet. The ivory scales contained in cases of mathematical instruments have the lines of equal parts usually divided by this mode; but as it frequently happens that the draughtsman wants a scale not found on these, it is necessary that he should understand the method and principle of constructing them for himself. (Note C.)

PROBLEM 5.

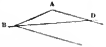
To bisect a given Angle.



1. Equal distances being set off from the angular point along the legs, from the points thus marked as centres, with any radius, describe two arcs cutting each other; then a line drawn from the angular point through the intersection will divide the angle into two others equal to each other. (Geo. I. § 7, Prop. 46.)

It is better to take a greater radius to describe the arcs than the distance between the points on the legs, as is directed to be done in the construction given in the treatises on geometry; because the farther the intersection of these arcs is from the angular points, provided the arcs do not cut each other too obliquely, so much the more accurately can the bisecting line be drawn.

2. Through any point A, taken at pleasure in either leg of the

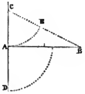


angle, but as far from the vertex B as may be convenient, draw a line A D parallel to the other leg; make A D on this parallel equal to A B; a line drawn from the angular point through D will bisect the angle. (Geo. I. § 3, Pr. 15, Cor. 2; and I. § 2, Prop. 6.)

If it be required to divide a given angle into more than two equal parts, it can only be done, practically, by describing an arc from the vertex as a centre, and dividing the part intercepted between the legs, by trial, into the number of parts required: the same observations as were made in the preceding problem, on dividing a line, being equally applicable in dividing an arc. (Geo. Schol. to Prop. 46.)

Problem 6.

To divide a given finite right line A B *medially, or in extreme and mean ratio.* (Geo. II. § 7, Def. 21.) *That is, to divide it so that the whole line shall be to the greater segment as this is to the other segment; or so that the rectangle under the whole line and the lesser segment shall be equal to the square of the greater.*



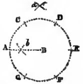
1. Draw a perpendicular at either end of the given line, as D C (Prob. 1.), and produced on both sides of it; bisect the given line, and set the half thus found along the perpendicular from A to C, take the hypotenuse B C and set it off from C to D in the perpendicular, the distance A D will be the greater segment, and may be set off from A or B along the given line. (Euc. 6, Pr. 30.)

2. The perpendicular A C being drawn, and made equal to half the given line, join B C, and set off A C from C to E in C B; the remainder B E will be equal to the greater segment. (Geo. II. Prop. 59.)

To produce a given finite line A B, so that it shall be one segment of the produced line divided medially. Divide the given line A B medially, and make the part produced equal to the greater segment of this division, A B will now be the greater segment of the whole line. And if to this produced line a part be again added equal to A B, the whole line now found will be divided medially, and A B will be the lesser segment. (Geo. II. Prop. 59, Schol.)

A distance A B can be divided medially by means of the compasses alone.

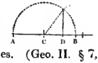
On A or B, with the distance A B as a radius, describe a circle, and step the radius round from A to C, D, E, F, G; with A D, the chord of two of these divisions, as a radius, from each end of the diameter A E, describe arcs intersecting in *a*; with the distance B *a* as a radius describe arcs from D and F, intersecting in *b*, then this point will be in the right line which would join A B, and will divide it medially as desired. (Note D.)



Problem 7.

To find a mean proportional between two given right lines.

Draw a right line A B, and make it equal in length to the sum of the given ones, and having the length of each determined by a point D in it, draw a perpendicular to the line from D, and bisect A B in C; from C, with the half of the whole line A C or C B as a radius cut the perpendicular by an arc; this perpendicular thus terminated will be the mean proportional to the two given lines. (Geo. II. § 7, Prop. 51.)

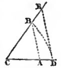


If the given lines are very unequal in length, the perpendicular in this construction will be near the end of the line, and the arc will necessarily cut it very obliquely, so that the true termination will not be well defined; to remedy this, set off the distance C D from D along the line produced, on the other side of the perpendicular, and from this new point, with the same radius as before, namely the half of the whole line, cut the former arc and the perpendicular more definitely in the required termination.

Problem 8.

To find a third proportional to two given right lines.

Draw two lines at pleasure, forming an angle of about 45°, and from the angular point C set off C A, C D on one of them, equal to the two given lines; and set off C B on the other, equal to C D or to the longest; draw D E parallel to the line which would join A, B, to cut the other leg in E, then C E will be the third proportional to C A, C D the given lines. (Geo. II. Prop. 52.)

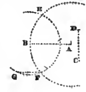


It is not necessary to draw the lines A B, D E; if a parallel rule be made use of, and set to the two first points, and then moved to the other, the dot or point on the second leg may be marked, without drawing the line.

It is, in the principle of this construction, immaterial what angle the lines make, but it will be seen that, in practice, if the angle was taken too acute, the parallels would be so short as not to allow of their being drawn with sufficient accuracy, and if the angle were too obtuse, these parallels would cut the legs too obliquely to define the intersections with sufficient precision; hence the lines are directed to be drawn at about 45° of inclination. (See also observations on Prob. 10, § 1.)

A third proportional to two distances, given by means of points, may be found by means of the compasses alone.

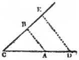
Let A B be one distance, and C D the other; from A with the radius A B describe an arc, and from B with the radius C D describe another, cutting the first in E, F; from E set off the radius C D three times to G, E G will therefore be the extremities of a diameter, then the distance F G, or the chord which may be supposed to join those points, is a third proportional to A B, C D. (Note E.)



PROBLEM 9.

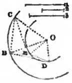
To find a fourth proportional to three given lines.

Draw two lines forming an angle of 45°, as in the last problem, and make C A, C D on one of them equal to the first and third of the given lines, and C B on the other leg equal to the second: then a line drawn through D, parallel to that which would join A and B, will cut off the segment C E equal to the fourth proportional sought. (Geo. II. Prop. 53.) See observation on Prob. 10.



A fourth proportional to three distances 1, 2, and 3, given by points, may be found by means of the compasses alone.

From any centre O describe two circles with 1, 2, the first and second of the given distances; and from any point B in the outer one, set off B C on its circumference, equal to 3, the third given distance; from B and C with any radius, B O for example, describe arcs cutting the inner circumference in D and E, then the distance or chord D E will be the fourth proportional sought. (Note F.)



By this problem, a line, A, being given, another, B, can be found, which will contain a rectangle equal to that contained by two other given lines C and D: for A : C :: D : B (Geo. II. Prop. 38, Cor. 1, 2, 3, 4), that is B is a fourth proportional to A, C, and D: this application is required in subsequent problems.

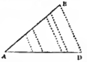
If three rectangles were given by means of three lines, each divided by a point into two segments equal to their sides; a line equal to the sum of the sides of a fourth may be found, the area of which will be in continued proportion to those of the given rectangles. Let A C B,* D F E, G I H, be three given rectangles, to which it is required to find a fourth proportional. First find (Prob. 7) a mean proportional between the two segments of each of the three given lines A B, D E, G H. Then by the present problem, find a fourth proportional to the three geometrical means thus obtained; these will be the sides of squares equal in area to the rectangles respectively (Geo. I. § 7, Prop. 58; and II. § 7, Prop. 51); and by Prob. 29, Case 4, this fourth proportional may be resolved into two segments of a line to which it is the mean, and which will be the sides of the fourth rectangle required. If one of the sides of this be given, the other will be determined.

By drawing an indefinite line with a perpendicular to it from any point, and then setting off the segments A C, C B; D F, F E, &c. on each side of the base of the perpendicular respectively, the process of finding the means will be much simplified.

Problem 10.

To divide a given right line into the same number of segments, and which shall be to each other in the same ratio, as those into which another line A D *is divided.*

1. From one extremity of the divided line draw a line A B, making an angle of about 45°, and equal in length to the given line which is to be divided, join B D the extremities of these two lines; and parallel to this last line draw others through the points of division in the divided line, or mark off the points where these parallels would cut the undivided one; these marks will divide this last into the segments required, and which may be transferred to the given line. (Geo. II. Prop. 54.)†



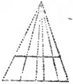
---

* By the rectangle A C B, &c. is meant a right line A B divided into two segments, A C, C B, equal to the sides of that rectangle; and so of the others.

† The following method of transferring any number of unequal lengths from one straight line to another will be found the best. Cut a strip of paper with a sharp knife, and by the edge of a straight rule; the edge of this is to be applied along the

This construction is best when the two given lines are not very unequal in length: but both in this and in the two preceding problems, though the two lines are recommended to be drawn generally at about an angle of 45°, to avoid the parallels intersecting them too obliquely, yet this angle must be modified, in order to fulfil this condition, according as the two lines are more or less unequal.

2. Draw a line, equal in length to the line to be divided, and parallel to the given one; draw lines through the extremities of the two thus placed, which will meet in a point, from which lines through the points of division of the divided line, which will cut the other in the proportional segments required; or mark off on the latter the intersections of such lines. (Geo. II. Prop. 30.)



This method is to be preferred when the lines are too unequal in length to admit of the first construction being used with advantage. It will easily be perceived that the farther apart the lines are drawn the farther off the point of concourse of those joining their extremities will be; and by so much the more accurate the construction, from the lines cutting the parallels less obliquely.

As any numbers, taken from two different scales of equal parts, will give lengths in the ratio of those numbers: by making use of two scales on which the two given lines shall contain exactly the same number of units respectively; this problem may be occasionally solved, by measuring the parts of the divided line from the scale belonging to it, and taking the corresponding numerical quantities from the other scale, and setting their distances on that line. (See following Problem.)

Cor. Hence, also, if a ratio be given by two numbers, two lines having the same ratio to each other may be obtained, by taking the given numbers from the *same* scale of equal parts.

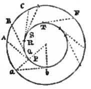
PROBLEM 11.

Any number of right lines being given, to find an equal number in the same ratio to each other respectively as those given ones; and having a given ratio to the first.

Describe two concentric circles, whose radii shall be in the

---

divided line, and marks made on the paper with a sharp pencil, very correctly opposite the points of division; the strip being then applied, so that the edge of it shall coincide with the line to be divided, the divisions are transferred from it by a pencil to that line. This will be found more expeditious and much neater than employing the compasses, and if carefully performed, quite as accurate. In linear perspective, as will be subsequently seen, it is indispensable.



given ratio (by Cor. to preceding Problem), and as large as may be convenient; Take any point in the circumference of each ($a . b$) at pleasure, so that the line joining them shall be, as nearly as may be, a tangent to the inner circle. From the point (a) on the outer circle set off on its circumference the chords equal to the given lines in succession (as a A, A B, B C); and from each of the points thus marked, with the constant distance ($a b$) mark on the circumference of the inner circle a succession of points (P, Q, R, S, &c.), then the chords of these segments of this inner circle taken first from the assumed point b, and then in succession, will be the lengths of the lines required (that is, $a\text{A} : b\text{P}$ in the given ratio, and $a\text{A} : \text{AB} : \text{BC}$, &c. $b\text{P} :: \text{PQ} : \text{QR}$.

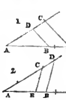
If the given lines were the sides of the triangles into which an irregular polygon were divided, and it were required to construct a similar one, the area of which should be in a given ratio to that of the first, the radii of the two circles being taken as the *square roots* of the terms of the given ratio, this problem would give the sides of the similar triangles respectively, of which the required polygon would consist. (Note G.)

For the method of finding, geometrically, the square roots of whole numbers from 1 to 10, and of decimals from 1· to ·01, see Problem 28.

Problem 12.

To divide a given right line A B, *either internally or externally, in a given ratio.**

1st. If the ratio be given by two lines—



From one extremity of A B draw a line to make any angle of about 45°, (vide observ. Prob. 10.), and set off on it, from the vertex, the length of the greater of the given lines of the ratio to C; from the termination of this, set off on the same line the length of the smaller line to D, back towards the angular point, if the given line is to be divided externally; and on farther from the angular point, if it is to be divided internally: from D draw a line to B, and through the point C, on the first leg, draw one parallel to it which will cut the given line, or that line produced in E in the required ratio. (Geo. II. § 7, Prop. 55.)

---

* A line is said to be divided *internally* or *externally*, when the line itself is divided into two segments; or the line is produced till the whole line so produced, and the original line, shall be in the proposed ratio.

2. If the ratio be given by two numbers—

Divide the given line into as many equal parts as is expressed by the sum of the numbers denoting the ratio brought to its lowest terms, if the division is to be internal; and the difference of those numbers, if the division is to be external. In the former supposition, the point of division will be at that of the equal divisions, which divides the line into the terms of the ratio; and in the latter, the line being produced, the number of the equal divisions must be continued for as many as is expressed by the smaller term of the ratio.

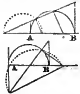
N. B. A ratio is brought to its lowest terms by dividing both terms by the greatest common divisor; thus 2 : 3 is the ratio 8 : 12 in its lowest terms.

The proportional compasses may be used, by setting the centre so that the longer legs may include the sum of the lines, or terms of the given ratio, when the shorter include the greater line or term; taking the terms from any scale of equal parts, if the ratio be given by numbers: then the given line being taken between one pair of legs, the other will give the segment, or the length of the line if it be to be produced, according as the division was to be internal or external, and consequently was taken between the longer or shorter legs.

Problem 13.

To divide a given right line A B, *either internally or externally, so that the rectangle under its segments shall be equal to a given rectangle.**

1. On A B describe a semicircle, and draw a perpendicular to A B at each end of it, which will therefore be tangents to the semicircle (Geo. III. Pr. 2.); and make these tangents equal in length to the two sides of the given rectangle: from either of the points, where the line joining the extremities of these tangents cuts the circle, draw a perpendicular to that line, and this perpendicular will cut the given line in the segments required. (Leslie, 6 El. Prop. 19. Geo. II. § 7, Prop. 56.)



If the sides of the given rectangle are so large, that when they are set off on the tangents the line joining their extremities will

---

* See preceding note.

not cut the circle at all, then one tangent must be drawn on the contrary side of A B to the first, and the respective lengths being set along them in these opposite directions, the line joining the extremities will necessarily cut the circle, and the perpendicular to this line will divide the given line, *externally*, in the required segments, as shown in the second figure.

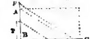
As a slight inaccuracy in drawing the perpendicular to the line joining the extremities of the tangents will cause it to cut A P very erroneously, this part of the construction will require great care, or the result should be verified, by drawing a line from the point of division in the given line, as found by this perpendicular, to one extremity of the tangent; this line being bisected, and a semicircle being drawn on it, will cut the first semicircle in the same point, if the right angle was drawn truly. Or the construction may be repeated, changing the tangents on which the sides of the rectangle are laid off: by this means a new point of division will be obtained in the given line, and if the corresponding segments of this double division are equal, the two operations were correct; if not, the process must be revived, or a mean taken between the two results.

If the given rectangle were a square, the two tangents will be equal; and in the first case, the line joining them will be parallel to the given line; and the perpendicular to it at either of the points where it cuts the semicircle will be obviously parallel to the tangents: in the second case, the line joining the tangents will pass through the centre of the semicircle, the constructions will in both cases be much simplified.

Problem 14.

To find two lines in a given ratio P : Q *which shall contain a rectangle equal to a given rectangle* A C.

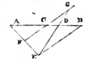
Make A B : B D :: P : Q. (Prob. 9.) Take B E, a mean proportional between B D and B C (Prob. 7.), and draw E F parallel to A D; then B F, B E will be the lines required. (Geo. II. Prop. 63. Euc. 6, Prop. 20, Cor. 2.)



A B, B C may be drawn at any angle; it will be better therefore in practice to draw them at about 45°, for reasons before given; in the figure they were shown perpendicular, to render the principle more apparent.

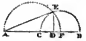
Problem 15.

To find an harmonical mean between two given right lines; that is, to find a line so that the greater of the two given ones is to the less, as the difference between the line found and the greater is to the difference between this third line and the smaller of them.



1. Draw any line A B and make it equal to the longer of the two given lines, from either extremity set A C off on it equal to the length of the shorter; take any point E at pleasure out of the line, and join E A, E B; through C draw a line parallel to E B to cut E A in F. Make C G equal to C F, join E G, cutting A B in D; then A D will be the harmonical mean required between A B and A C. (Note H.)

2. A B, A C being made equal the given lines as before, divide B C the difference between them in D, so that B D : B C :: A C : A B (Prob. 8, 13.), that is, divide B C in D in the ratio of A B to A C; then A D is the harmonical mean sought, as before. (Geo. II. § 7, Prop. 57.)



3. Let A B, A C be the two given lines as before: on B C the difference between them describe a semicircle B E C, or as much of an arc of one as may be necessary, and let F be the centre of it. Then on A F, as a diameter, describe another semicircle, or an arc, cutting the first in E, and draw D E perpendicular to A B.

Then A D will be the harmonical mean between A C and A B and A B : A C :: B D : C D. (Geo. App. Lemma, Prop. 16, 17.)

This construction is preferable to either of the preceding; and if the arcs of the circles are marked on the opposite side of the line, the perpendicular may be drawn through the point of their intersection at once.

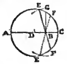
If two lines A C, A D were given, and it were required to find a third harmonical progressional (Geo. II. § 6, Def. 17.) to them, the following constructions on the same principles must be made. (See the first diagram.)

1. Through C draw any line at pleasure (at about 45°) and set off any distance both ways, from C on it to F and G; draw two lines, one from A through F and one from G through D, meeting in E; through E draw E B parallel to F G to meet A D produced in B; then A B will be the third harmonical progressional to A C, A D.

2. Or divide A D produced in B, so that A B : B D :: A C : D C, (Problems 8, 13.) that is in the ratio of A C to C D. (Geo. II. § 7, Prop. 58.)

The first of these constructions is objectionable, from the difficulty in practice of determining with accuracy a point, from the intersection of prolonged lines; it should therefore only be used when the given lines are more than two inches long at least; the other, though longer, is preferable when great correctness is required. On some occasions the following construction may be found useful.

Let A B, B C be the two given lines placed in one continued right line; bisect A C, their sum, in D, and describe a circle on A C as a diameter. With the same radius A D, from B as a centre, intersect the circumference in E, and draw E F through B; also draw the perpendicular B G.

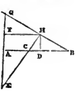


Then E B = A D is the *arithmetical mean*, B G the *geometrical mean*, and B F the *harmonical mean* between the two lines A B, B C. (Geo. II. Prop. 51; Prop. 47, Cor. 2; and Geo. III. Prop. 20, Cor.)

PROBLEM 16.

To divide a given line A B *in harmonical proportion, and in a given ratio; that is, to find two points* C D *in a given line* A B, *so that* A B : A C :: B D : C D :: *the one term* : *the other, of the given ratio.*

Through either end of AB draw a line EG at pleasure, at about right angles to it, and make A E, A G each equal to the larger term, and A F equal to the smaller term of the given ratio (Prob. 10, Cor.) and join B G: through F draw a line parallel to A B, cutting B G in H, then a line joining E H will cut A B in C one point; and another through H parallel to E G will cut A B in D, the other point of harmonical division of A B. (Leslie, El. 6, Prop. 6.)

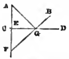


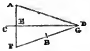
The harmonical division of lines is of frequent application in linear perspective.

PROBLEM 17.

From two given points A, B, *to draw lines to meet in some point of a given right line* C D, *so that they shall make equal angles with it.*

From either of the given points A, draw A F perpendicular to C D, cutting it in E; make E F equal to E A; from B, the other point, draw B F, cutting C D in G; join A G; then A G B G will make equal angles, as desired, with C D. (Note I.)

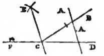




PROBLEM 18.

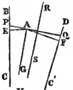
To draw through a given point A, *a line which shall make equal angles with the legs of a given angle* B C D.

Bisect B C F, the supplemental angle to B C D, by a line C E (Prob. 5.); then a line drawn through A, parallel to C E, will form equal angles with C B, C D as required. (Geo. I. Prop. 15.)



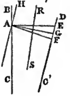
If the angular point C do not fall within the drawing, that is, if the point of convergence of two lines B C, D C′ be inaccessible, a line to make equal angles with them may be drawn by the following construction:—

1. From A, the given point, draw A E, A F at right angles to the given lines, bisect the angle E A F, formed by the perpendiculars, by the line A G; then P Q, drawn through A at right angles to A G, will make equal angles with B C, D C′.



2. If A be in one of the lines, the angle E A F formed by the perpendiculars to the given lines being bisected by the line A G as before, this last will make equal angles with B C, D C′.

Or if through A a line A H be drawn parallel to D C′, then A G must be drawn to bisect the angle C A H.

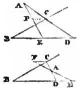


If the line thus found A G (P Q) be bisected by a perpendicular R S, and a line be drawn by the above constructions to make equal angles with either of the given lines and R S, this will be another chord of a circle described from the point of convergence as a centre to pass through P Q or A G.

These constructions are of use to masons and carpenters for finding the mould to cut the arc of a stone forming part of a large arch.

Problem 19.

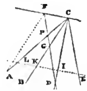
Through a given point A *to draw a right line, so that the segments intercepted by the legs of a given angle* C B D *may be to each other in a given ratio.*



From A, with any radius, cut the farther leg in the point E, join A E, and make A F : A E in the given ratio, (Prob. 10. Cor.); draw F C parallel to B D to cut B C in C; then a line drawn through A and C will be the line required. (Geo. II. Prop. 61.)

Problem 20.

Given four lines, C A, C B, C D, C E *meeting in a point* C, *to draw a line to cut them all, so that the segments intercepted by the two exterior pair shall be equal.*

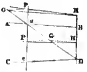


Through any point P, in either of the exterior lines, C A, draw P D parallel to the other exterior line, C E; make P F in P D produced a mean proportional between P G and P D, (Prob. 7.) join F C; then any line parallel to F C will cut the given lines as required. (Note J.)

It is obvious that the farther from C, P is taken, by so much the more accurate will be the construction.

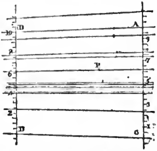
Problem 21.

Given two lines A B, C D *converging to an inaccessible point; to draw a line through a given point* P, *which would, if produced, pass through that inaccessible point.*



Through P draw a line nearly at equal angles to the given lines, cutting them in *a* and *c*, and draw the indefinite line D H parallel to *a c*, as much nearer the point of convergence as possible, cutting C D in D; join *a* D, and through P draw P G parallel to C D, cutting *a* D in G; from G draw G H parallel to A B, cutting D H in H, then P H being drawn, it will be the line required. (Note K.)

The foregoing is the best construction when only one or two lines are wanted; but when many are required through different points, the shortest method is to draw two parallel lines as far apart as the drawing will admit of, intersecting the given lines nearly at equal angles, and divide both segments between the converging lines into the *same number* of equal parts, very carefully, of not more than a quarter of an inch each, on the longest line; the divisions may be continued each way beyond the converging lines, and may be numbered at corresponding divisions*. A rule applied to any points through which the lines are to pass, is to be adjusted by trial so that it shall apply also to the same number of divisions from either given line; and if no two corresponding divisions are found to lie in the same line with the point, the spaces between the two nearest divisions must be subdivided by the eye proportionally, so that the line drawn by the edge of the rule through the given point may pass through the same proportional part of the two corresponding divisions of the divided lines: for example, the line passing through P would not coincide with divisions 6, but is drawn so as to divide the spaces from 6 to 7 on each line into proportional parts, as one-fifth of each, and in the same manner the other lines are drawn.

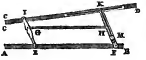


If the drawing be to a large scale, and many converging lines very near each other are required, as is the case with cornices, or any set of mouldings in architectural drawings in perspective, it will be necessary to subdivide the divisions through which they will pass into 5 or 10, for the sake of greater nicety, and these subdivisions must be proceeded with as the larger were for more remote lines; or the proportional compasses may be set to the two divisions on the two scales, and thus any part of one division may be set off proportionally on the corresponding one on the other divided line. (Geo. II. § 4. Prop. 30.)

---

* In the figure, the alternate divisions are only numbered, to avoid confusion.

The *centrolinead* is an instrument invented by Mr. P. Nicholson, senior, for the purpose of drawing lines tending to the inaccessible point where two given lines would meet.



It consists of two rules, A B, C D, connected by two bars, E I, F K; the shorter E I is simple, like that which connects the two rules of a common parallel rule; the other bar, F K, consists of two parts, one above the other; to the lower one, and to E I, a third rule G H is jointed, so that G H being eqnal to E F, and G E to H F, by the motion on the four centres the figure G E F H always forms a parallelogram, on the same principle as the common parallel rule. On the lower part of F K two pins are fixed, which pass through a longitudinal groove in the upper part, and thus the compound bar is kept in a right line, and admits of being lengthened or shortened for adjustment; there is a screw at M, by which the parts are clamped together when this is made: the upper part of F K is attached to the rule C D, by a joint carried on a moveable piece which slides in a dovetailed groove of that bar.

To use the instrument, set A B to one of two converging lines, the screw M being loose; move the bar C D till it coincide with the other; then clamp M, and by the compound motion of the centre K in the groove, the rule C D will always tend to the point of concourse of the two original lines. (Note L.)

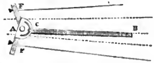
The centres I K are so contrived, that they are in the line of the edge of C D, but those E F being in the middle of A B, two corresponding marks in the line of them are made on the bevelled ends of this rule, and these marks must be set to the first original line. This construction is essential to the correct action of the instrument.

As with the parallel rule, great care must be taken while using the centrolinead that A B does not alter its position.

The principle of the wheel parallel rule admits of being applied to a centrolinead, if the rollers are made of unequal diameters, and can be moved along the axis for adjustment; but such an instrument would be troublesome to use, and very liable to shift on the paper.

Another instrument for the same purpose is the invention of Mr. John Farey.

A B is a rule like the blade of a T square, E and F are two shorter and stouter pieces; one carries at A an arbor, on which A B and the other part can be fixed, so that all three shall have a common centre of motion, which is in a line with the *drawing edge* of A B and the back or outward edges of E and F. There is a clamping screw attached to the blade, which, by means of small brass arcs carried by the *stock* pieces, fixes the three parts at the proper angles when the instrument is adjusted.



Two small *fulcra*, with knife edges, are fixed by points at the bottom of them to the drawing-board, so that these edges shall be in the given converging lines. Against these the back of the stock pieces slide, and the blade must be fixed to the latter by trial at the proper angle, so that it shall coincide with the two given lines when brought to them in succession. When this is the case, and the pieces are clamped together by the screw, lines drawn by the edge of the blade will tend to the same point of convergence as the two given lines.

The difficulty of adjustment, and the liability of the fulcra to be deranged by the pressure of the stock against them, are the chief objections to this instrument, which is otherwise preferable to the former, from being cheaper.

The principle on which it acts, and the construction of a simpler instrument for the same purpose, is to be found in the note at page 236 of the Treatise on Geometry.

PROBLEMS RELATING TO RIGHT LINE FIGURES.

PROBLEM 22.

To construct a triangle from three or more parts given.

Case 1. Given the three sides.

Draw a line, and make it equal to any one of the given sides, from each extremity, with radii equal to the two other sides; describe arcs to cut each other, and lines being drawn from the intersection to the two ends of the first line, the triangle will be completed. (Geo. I. § 7. Pr. 50.) If the sides are given by numbers, they must be taken from a scale of equal parts.

Case 2. If two sides and an angle be given.

1. Make an angle equal to the given one; and if this be the angle included by the two given sides, make its legs equal to those sides, and join the extremities of them for the third.

2. But if the given angle be not the included one, then make one of the legs of the constructed angle equal to one of the sides, and from the point thus marked, with the second given side for a radius, cut the other leg of the angle in the third point of the required triangle.

Case 3. Given one side and two angles.

1. Draw a line equal to the given side; then if the given side be the one interjacent to the angles, at each extremity of the constructed side make an angle equal to one of the given ones respectively, and the legs of these angles will cut each other, and form the two remaining sides of the triangle.

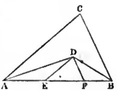
2. But if the given side be not the interjacent one, then construct at one end of the line, drawn equal to the given side, one of the given angles, and at the other end an angle equal to the difference of the sum of the two given angles, and two right ones, or 180°. (Geo. I. § 7. Prop. 50.) The legs of these angles will form the required triangle, as before.

It is frequently advisable, when great accuracy is required, to calculate trigonometrically the remaining sides from the data, and then to construct the triangle from the three sides, that being the method least liable to error.

No specific triangle can be constructed, if the three angles only are given, since these data are common to an infinite number of triangles. In this case, the problem resolves itself into constructing a triangle *similar* to a given triangle.

As a line parallel to any side of a triangle will, by meeting the other two sides, or those sides produced, form a new triangle similar to the first: (Geo. II. § 3. Prop. 29.) a triangle, similar to a given one, that is, having its three angles equal to those of a given one, may be immediately obtained by drawing such a parallel to any side of it.

Case 4. Given the three angles, and the perimeter or sum of the three sides.



1. Draw a line A B, and make it equal to the given perimeter, and at each extremity make an angle B A C, A B C equal respectively to two of the given angles; bisect these by A D, B D meeting in D, from the point D draw D E, D F parallel to A C, B C, then D E F is the triangle sought. (Geo. I. Prop. 51.)

Problem 23.

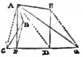
To construct a triangle equal to a given rectilinear figure, that is, having the same area.

Case 1. To construct a triangle equal to any given parallelogram.

Draw two lines parallel to each other at the same perpendicular distance, that two opposite sides of the parallelogram are, and make one equal to twice the length of one of those sides, for the base of the given triangle, the vertex of which may be taken in any point of the other parallel, consequently the triangle may be made with one angle of any given magnitude. (Geo. I. § 4. Prop. 25, 26, 27.)

By the converse of this, a rectangle may be made equal to a given triangle, by taking the base and half the perpendicular height; or half the base and the perpendicular height of the triangle for the sides of the rectangle. (Geo. I. § 4. Prop. 26, 27.)

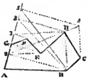
Case 2. To construct a triangle equal to any given rectilinear polygon.



1. Through any angular point B of the given polygon A B C D E, draw B P parallel to the line A C, which would join the two adjacent angles A and C, and let B P cut any side or side produced in P; join A P or C P according as P is situated; through E draw a line parallel to that, A D, joining the two angles adjacent to E, and cutting the same

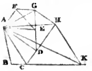
side or side produced in Q; join A Q; and the triangle A P Q will be equal to the given figure. (Geo. I. Prop. 54; and Leslie, El. II. Prop. 5.)

2. Let A B C D E F G be the given figure.



Produce A G, one side, indefinitely; through F draw a line parallel to that joining E G, the adjacent angles, to cut A G in 1; through E draw a line parallel to D 1, cutting E G in 2; through D draw a parallel to B 2, cutting E G in 3; through C draw a parallel to B D, to cut D 2 produced in 4; then through 4 draw a parallel to B 2, or D 3, cutting A G in 5; and B 5 being joined, A B 5 will be the triangle equivalent to the original polygon.

3. Let A B C D E F be the given figure.



From A draw lines to the other angles of the figure; through F draw F G parallel to A E, to meet D E produced in G; through G draw G H parallel to A D, to meet C D produced in H; through H draw H K parallel to A C, to meet B C produced in K; then A K being joined, the triangle A B K will be equal to the original figure. (Geo. I. Prop. 54.)

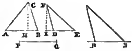
By the two last, the equivalent triangle is obtained with one side A B, and an adjacent angle in common with the original polygon. If this have a still greater number of sides, the same principle of proceeding must be continued, observing that the object always is to obtain an additional triangle applied to one side of that previously obtained, and equal to each successive triangular portion of the original; but so much depends on the form of the polygon, that no rule can be given as to the point from which the constructions must commence: if possible, however, the longest side should be chosen to reduce the triangles to, and so that the resulting one may be not very obtuse-angled. In all cases, the constructions must be very carefully made, as a slight error in any step may cause a considerable one in the final triangle. If the given polygon be very complicated, it had better be divided into two or more by diagonals, and these proceeded with; then the areas of the resulting triangles may be added together, or the figure may have some of its irregular portions reduced to some one or two sides of it by the problem, and then the figure thus simplified proceeded with as above.

This problem is of great use to surveyors, who by means of it can obtain a triangle equal to the irregular polygon representing a plot of ground, and can thus more readily ascertain its area.

PROBLEM 24.

To construct a triangle equal to a given rectilinear figure, and similar to another given triangle D E F.

Case 1. If the given figure be a triangle A B C :



Draw a perpendicular from any angle to the opposite side in each triangle, as C H, F I; find P Q a fourth proportional to F I, C H, and A B, the base of the given triangle to which the required one is to be equal. (Prob. 9.) Make R S a mean proportional between D E and P Q, (Prob. 7.) and on R S, as a base, make the triangle R S T similar to D E F, (Prob. 22.) and it will be equal to A B C. (Euc. 6. Prop. 15, 20. Cor. 2, 25, and Geo. I. Prop. 26.)

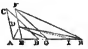
The construction will be simplified by setting the distances to obtain P Q along the sides of either triangle produced; and when R S is found, it may be set off from D on D E, or D E produced, and a line drawn through the end parallel to E F to meet D F, or D F produced; by this a triangle similar to D E F will be made.

Case 2. If the given figure be not a triangle:

A triangle equal to it must be found by the preceding problem, and then, by Case 1. of this, another equal to it, and similar to D E F, can be constructed.

PROBLEM 25.

On a side, or a side produced, of a given triangle A B C, *to construct another equal to it, that shall have a given point* D *for its vertex.*



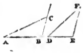
Through D draw a line parallel to either of the other sides of the triangle, as A C, cutting the base in E; and make E F equal to A C, and E G equal to A B; join F G; the triangle E F G will therefore be equal and similar to A B C: through F draw a line parallel to D G, cutting the base, or this produced in H. Join D H, and the triangle E H will be that required. (Geo. I. Prop. 27.)

If A I be made equal to E H and D A, D I joined, then the triangle D A I, equal to D E H, will be equal to A B C, and have D for its vertex.

Problem 26.

On a given line A B *for a base, to construct a triangle equal to a given triangle* D E F.

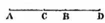
At either extremity of A B make an angle A B C equal to one of those of the given triangle D E F, and make B C a fourth proportional to A B, D E, E F (Prob. 9); join A C, and the triangle A B C will be equal to D E F. (Geo. II. Prop. 41.)



Problem 27.

On a given line, A B, *for a base, to construct an isosceles triangle, having each angle at the base double that at the vertex.*

Divide A B medially in C (Prob. 6), and produce it till the part B D produced is equal to the greater segment A C; then the whole line A D will be the side of the required triangle. (Leslie, El. IV. Prop. 3 and 4.)



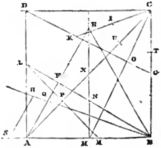
Problem 28.

To find the sides of the squares, the areas of which shall be any part or multiple, from 1 *to* 10, *of that of a square on a given line* A B.

1. On A B describe the original square A B C D. Draw the diagonal A C, and bisect it in X; bisect the sides AB, BC in H and G.

Then A B being unity, A X will be equal $\sqrt{\frac{1}{2}}$, and B G will be $\sqrt{\frac{1}{4}}$.

On A B construct the equilateral triangle A B E, bisect either side A E in F, draw B F produced to cut A D in L; then B N, or half B L, will be $\sqrt{\frac{1}{3}}$.



Join CH, D G cutting in O; then C O will be $\sqrt{\frac{1}{5}}$.

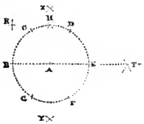
Make A M equal to B N or A L; join L M, cutting A C in P; then A P will be $\sqrt{\frac{1}{6}}$.

Make A Q equal to $\frac{1}{3}$ of A E of A B; draw B R through Q, and make it equal to A B: draw R S parallel to A E, to cut A B produced in S; then R S will be $\sqrt{\frac{1}{7}}$.

Bisect C X, the semidiagonal, in U; then C U, or $\frac{1}{4}$ of the diagonal, will be $\sqrt{\frac{1}{8}}$: and C T, being made equal to $\frac{1}{3}$ of B C (A B), will be $\sqrt{\frac{1}{9}}$.

Bisect D O in K, and join KC; and this last line being again bisected in I, CI will be $\sqrt{\frac{1}{10}}$.

That is, the areas of squares on A X, B N, B G, C O, A P, R S, C U, C T, C I, will be $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{7}$, $\frac{1}{8}$, $\frac{1}{9}$, $\frac{1}{10}$ of A B C D.



2. With A B for a radius, describe a circle, and step the radius round to C, D, E, F, G, with B D, the chord of two of these divisions, describe arcs from B and E, the extremities of the diameter, intersecting in X and Y; and with the same radius B D, from D and F, describe arcs intersecting in T: with A X or A Y, from B and E as centres, bisect the semicircle in H, and with A B for a radius from B and H, describe arcs intersecting in R; then A B being unity, A X will be $\sqrt{2}$; B D will be $\sqrt{3}$; B E, the diameter, will be $\sqrt{4}$; E R, $\sqrt{5}$; T X or T Y will be $\sqrt{6}$; C T, $\sqrt{7}$; X Y, $\sqrt{8}$; B T, $\sqrt{9}$; and T R will be $\sqrt{10}$.

That is, squares on these lines will be 2, 3, 4, &c... ten times the area of the square on AB. (Note M.)

The last construction, requiring the compasses alone, was given, because each quantity found being greater than the original line, the most accurate mode of proceeding is required to avoid error; but it is obvious that the solution may be made by right lines, as the former was: in this case it is worth remarking, that B A, A E, E T, are all equal; and that A D T will be a right angle, and that H will of course be in the line X Y, at right angles to B E, and passing through A.

If the original line be small, it will be better to double or treble it; and the construction being made with this, the resulting lines must be divided into two or three parts for the line sought.

Problem 29.

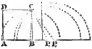
To construct a parallelogram equal to a given rectilinear figure.

Case 1. If the given figure be a triangle, (see Problem 23,) a rectangle may be made equal to it; and all parallelograms on the same base, and between the same parallels being equal, that required can be made with any angle. (Geo. I. § 4, Prop. 27.)

Case 2. Any irregular polygon can be reduced to a triangle by Prob. 23, and this converted into a rectangle by Case 1.

Case 3. A square equal to a given rectangle is constructed by making its side a mean proportional between the sides of the rectangle. (Prob. 7.)

Case 4. If any point P be taken in a side, or a side produced, of a square, A C, and a semicircle be described from it, with P C for a radius, the segments into which B divides the diameter will be the sides of a rectangle equal to the square: and if one side of this rectangle be required to be of a given length, this must be set off from B along the side A B or A B produced, and the semicircle described as before, to pass through the point thus marked, &c.



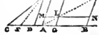
A square can be drawn with great readiness by laying a rectangular protractor scale on the paper, and drawing two lines by the parallel edges at once; then setting the scale at right angles to these lines, and drawing the two other sides by the same edges; the four sides are thus certain of being equal.

Problem 30.

On a given line, AB, *to describe a parallelogram equal to a given rectilinear figure.*

Case 1. If the given figure be a triangle, C D E,

Bisect C D in F, and join E F; make AG equal to C F, and through A, G and B draw parallel lines either perpendicular or making any given angle with A B: draw I K parallel to A B, at the same perpendicular distance from it that E is from C D, and join A K, cutting GH in L; a line through L parallel to A B will form the parallelogram A M N B, equal to the triangle C D E. (Geo. I. § 7. Pr. 57.)



Case 2. Any other right-lined figure can be reduced to a triangle by Prob. 23, and a parallelogram constructed by Case 1, equal to this triangle.

PROBLEM 31.

To find the side of a square equal in area to any number of given squares added together, or to the difference of two given squares.

1. Let A, B, C, D, &c. be the sides of the given squares:

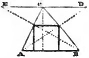
Draw a perpendicular at one extremity of a line equal to A, and make it equal to B; draw the hypotenuse to complete the right-angled triangle; and at one end of this hypotenuse draw another perpendicular, which make equal to C; complete this second right-angled triangle, and proceed as before till all the given sides are employed. The hypotenuse of the last triangle will be the side required of the square, equivalent in area to those on A, B, C, D, &c. (Geo. I. § 7. Prop. 60.)

2. Let A and B be the sides of two unequal squares, of which A is the least:

At one end of A draw a perpendicular, and from the other extremity, with a radius equal to B, describe an arc, cutting the perpendicular. The length of the perpendicular to the intersection will be the side of the square equal to the difference of A^2 and B^2. (Geo. I. § 7. Prop. 59.)

PROBLEM 32.

To inscribe a square in a given triangle,* A B C; *so that one side of the square shall coincide with, or form part of, one side of the triangle.*



Through any angle of the triangle C, draw a line parallel to the opposite side A B; draw a perpendicular to the same side from C; set off the length of this perpendicular both ways from C to D and E, and from the extremities of the base A and B draw lines to D and E, cutting the sides of the triangle in two angular points of the square: the side joining them will be parallel to A B, and the perpendiculars being drawn will complete the square required. (Leslie, An. I. Prop. 7.)

---

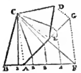
* A right line figure is said to be inscribed in another, when the points in which the sides of the former meet, are in the sides of the latter.

Problem 33.

To divide a given rectilinear figure into any number of equal parts by lines drawn through a given point.

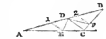
I. If the point be one angle of the figure.

1. To divide a triangle into equal parts: divide the base or side opposite the angular point into the required number of equal parts, and draw lines from the angle to the points of division, and the triangle will be divided by them as required. (Geo. I. § 4. Prop. 27.)



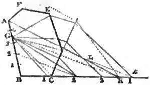
2. Any rectilinear figure, A B C D E, of more than three sides, may be divided into equal parts by lines drawn from any angle, as C, by first finding a triangle C B F equal to it, and having a side and angle common to the original, (Prob. 23). Then divide B F, the side of that triangle, into the required number of parts, for example five, in the points 1, 2, 3, 4. From such points of division, as 1, which fall within the given figure, draw lines to C, and from the others draw lines parallel to A C or F G, which will meet A E or A E produced: from those points where they meet A E, draw lines to C; but from those which lie between E and G draw new lines parallel to E C or G D, which will meet E D or E D produced; from those which lie in E D draw lines to C, and proceed with the rest as before, till the division of the figure is completed, by the lines drawn to C from the respective points of division in B F transferred to the sides of the original polygon. (Geo. I. § 7. Prop. 55. Cor. 2.)

II. If the given point be in one side of the figure.



1. To divide a triangle A B C into any number of equal parts, for example three, by lines drawn from a point D in A B, one side of it; divide A B into the required number of parts at 1, 2, and join D C; then draw 1 E, 2 F, parallel to D C, cutting A C, B C in E and F; lines drawn from D to E and F will divide the triangle as required. (Note N.)

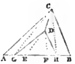
2. Any rectilinear figure A B C D E F may be divided into any number of equal parts, for example five, by lines drawn from a point G in one of its sides A B, by constructing a triangle A B K equal to it, and having the side A B in common (Problem 23). Then divide A B K into the required number of



parts by lines drawn from G, as G 1, G 2, G 3, G L:* of these, G 1 falling within the given figure will divide it into one of the required parts. (Geo. I. § 7. Prop. 56, Cor. 1.) From the points 2, 3, which lie between C and K, draw parallels to C G to meet C D, or C D produced. From the point where the parallel from 2 meets C D draw a line to G, for the second line of division of the figure; then from the point in which the parallel from 3 meets C D *produced*, draw a parallel to G D to meet the side D E, and from the point of intersection draw the third line of division to G. From the point L, which lies in the side A K of the equivalent triangle, draw a parallel to G K to meet B K produced in I, and join G I †; from I draw a parallel to G C to meet C D produced, and from the point thus found draw a parallel to G D. If this parallel meet D E, as in the figure, a line from the point in D E to G will complete the construction; but if the parallel meet D E produced, a new parallel to G E must be drawn to transfer the point to the side E F, as before. (Geo. I. § 7. Prop. 56, and Cors.)

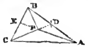
III. If the given point be within the figure.

1. To divide a triangle A B C into two, or into three equal parts, by lines from the point D.



Trisect the base A B, and draw D E, D F from the points of division; through C draw C G, C H parallel to these to cut the base in G and H; then D C, D G, and D H being joined, these lines will divide the triangle into three equal parts.

If the base A B had been bisected by a point P, and D P joined, a line drawn through D and the point in the base, where the parallel to D P through C would cut it, would bisect the triangle. (Leslie, An. 1, Pr. 6.)



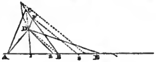
If it were required to find a point within a triangle such, that lines drawn from it to the three angles would divide the triangle into three equal parts, any two sides, as A B, B C, being bisected in D and E and A E, C D drawn, the point F where they intersect will be the point required. (Leslie, An. 1, Prop. 5.)

---

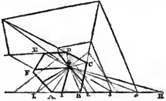
* To avoid the confusion so many lines would create, the construction by which those to divide the triangle are found, is only shown for the point 2; for the same reason G 3, G L, G 1, G E are not drawn.

† On account of the parallels G K, L I, the triangle G L K is equal to G K 4; consequently the whole triangle G L 3 is equal to the quadrilateral G 3, K L.

A triangle is divided into more than three equal parts by lines from a point D within it, by first constructing a triangle A D E equal to the given one A B C; having the given point D for its vertex (Prob. 25.). Divide A E, the base of this triangle, into the required number of parts, for example four, in 1, 2, 3 from such points as 1, 2 which fall within A B, draw lines to D. (Geo. I. § 7. Prop. 55, 56, Cors. 1.) Through the point 3 draw a parallel to D B, which will cut the side B C in a point 3′; join D 3′, then D A, D 1, D 2, D 3′, will divide A B C into four equal parts. If the parallel from 3 had cut B C produced, the point of intersection must have been transferred to A C, by drawing a parallel to D C through the point in B C produced; from where this parallel cuts A C a line drawn to D completes the division: this construction is shown in the figure, from which it will be understood without further explanation.



2. Any rectilinear figure can be divided into any number of equal parts, by lines drawn from a point G within it, by finding a triangle G I H equal to the given figure, and having the given point for its vertex, (Prob. 23 and 25.) Divide the base of this triangle, I H, into the required number of parts, for example five, in the points 1, 2, 3, 4; draw a line from G to 1 for one of the dividing lines, the point 1 falling within the figure; from the point 1 draw a parallel to G A, and from the point in which it cuts the side A F draw another of the dividing lines to G. From the points 2, 3, 4, draw parallels to G B, cutting the side B C or B C produced; from these points of intersection, which fall in the side B C, draw lines to G; from those which lie beyond C draw parallels to C G to meet C D or C D produced, from the points of intersection which lie in C D draw lines as before to G; from the others draw parallels again to G D, and proceed in the same way; the lines drawn from the points thus found on the sides to G will divide the figure into five equal parts.



As several of the constructions in this Problem are rather complicated, *all* the lines required are drawn in the diagrams, where confusion would not be produced by so doing; but the student will easily comprehend the mode of proceeding in other cases from what has been here given, if he distinctly keep in his mind that the principle chiefly employed is, that triangles on the same base and between parallel lines are equal (Geo. I. § 4. Prop. 27.); and that consequently a triangle equivalent to another can always be found, by drawing a parallel to the base through the vertex; and therefore having another line for a side, which is cut by that parallel in the vertex of the equivalent triangle sought. (Note O.)

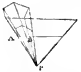
PROBLEM 34.

To construct a figure similar to a given figure.

Case 1. To construct a rectilinear figure similar to a given one, and on a given line A B.

1. Divide the original figure into triangles by diagonals, drawn from any one of its angles. Then on the given line make a triangle similar to that of the original, which contains the side to which the given line is intended to correspond; npply to the side of this a new triangle, similar to the next and corresponding one of the original, and so on till the figure is completed.

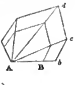
2. Place the given line A B parallel to the side of the original to which it is intended to be homologous; as they are not equal, lines drawn through their extremities will meet in some point P, from which lines must be drawn to all the other angles of the original. From each extremity of the given line draw lines parallel to the sides adjacent to the side of the original to which A B was placed parallel; where these lines meet those drawn from P, to the extremities of the sides of the original draw the next sides of the copy parallel to the corresponding ones of the original, and so on till the figure is completed. (Geo. II. § 7. Prop. 65.)



If it were required to construct a figure similar to a given one, and having its perimeter equal to a given line.

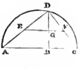
Produce the side of the given figure, and make the whole line equal to its perimeter. Draw another line parallel to this produced side, eqnal to the given line, and join the extremities of these parallels by lines, which will meet if produced in a point P; then by the above constrnction make the figure similar to the original, and it will have its perimeter equal to the given line. (Geo. II. § 7. Prop. 66.)

3. Set off the given line A B from one end A of the side A *b* of the original from A to B, and from A draw diagonals to the other angles of the figure, as A *c*, A *d*, &c., then through B draw a line parallel to *b c*, the adjacent side to meet the first diagonal A *c*, and from the point of meeting draw a line to the next diagonal A *d* parallel to the side *c d* of the original, and so proceed till the figure is completed. (Geo. II. § 7. Prop. 65.)



The following method of constructing a parallelogram similar to a given one, and having its area in a given ratio to that of this given one, will be found frequently of use.

Let the segments A B, B C of any right line at pleasure be made in the ratio of the areas of the two figures; on A C describe a semicircle, and draw B D perpendicular to A C, to cut the circumference in D; join A D, D C; make D E equal to a side of the given parallelogram, and draw E F parallel to A C, cutting D C in F; then D F will be the homologous side (Geo. II. § 4. Defin. 15.) of the required parallelogram: the other side might also be set off on D A, and a parallel to A C would in the same way determine on D C the second side of the figure sought. (Note P.)



The proportional compasses greatly facilitate the solution of this problem; adjust them so that the points at one end shall include the side of the original figure, when the points at the other extremity include the given line A B: as the given figure may be divided, by diagonals, into triangles, each triangle can be copied with the greatest accuracy and readiness, and every point of the copy can be verified by other diagonals without the necessity of drawing any lines but the perimeter of the copy, when all the points are found. By means of these instruments, or one of the foregoing constructions, any figure consisting of right lines, or arcs of circles only, can be copied to any scale or proportion to the original.

Case 2. If the original figure consist of curved lines not arcs of circles, or be so complicated that it could not be easily reduced or enlarged by the foregoing methods, recourse is generally had to the following mode of proceeding.

A rectangular parallelogram must be drawn to include within it the figure to be copied; taking every precaution that its sides shall be truly parallel and vertical to each other. Equal distances are to be stepped along all four sides, commencing from each end of two of them; these points of division being joined by lines parallel to the sides of the rectangle, its whole surface will be divided into small squares.

A *similar* parallelogram (Geo. II. § 4. Def. 14.) is next to be drawn on the paper on which the copy is to be made, equal to the former if the copy is to be of the same size, scale, or area as the original; but if not, greater or less in the proportion the copy is required to be. The sides of this are to be divided into the *same number* of equal parts that those of the first parallelogram were divided into, so that, by joining the points of division on the opposite sides by parallel lines, the surface may be divided into the *same number* of equal squares that the first contained; and the squares in the two will be obviously in the same proportion to each other that the parallelograms are. No trouble must be spared to ensure the utmost accuracy in constructing these parallelograms and the squares in them, as the copy will be otherwise incorrect. The spring dividers must therefore be always employed, and a very sharp and hard pencil and a very straight rule to draw the lines.

The corresponding squares will be most readily found by numbering the points of division along two sides in each figure; by means of these numbers the eye is guided to the squares very easily.

The copy is now to be made, by drawing by hand, or with a rule, all the lines or forms which are found in every square of the parallelogram on the original in the corresponding squares of the other, and if these lines are drawn with a pen and Indian ink, the pencil squares may be rubbed out when the copy is completed, without any injury having been produced by them, provided they were drawn very lightly and neatly.

It will be readily comprehended, that the size of the squares, into which the original parallelogram is divided, must be regulated by the subject to be copied: if the parts of this are minute, or if the lines are very irregular, the squares must be proportionally smaller; and as these conditions may vary indefinitely, no rule can be given; the artist must be guided by circumstances, and by his power of copying by eye, with accuracy, the parts comprehended in each square.

But the size of the squares on the original being determined from these considerations, the squares on the parallelogram, on which the copy is to be made, are entirely governed by the proportion in which the figure is to be copied.

1. If the copy is to be equal in size to the original, the squares must necessarily be equal to those on the original.

2. If the copy is simply required to be on a smaller or greater scale, or to be brought within a smaller or greater sheet of paper, the sides of the parallelogram must be taken accordingly, only

taking care that they shall be in precisely the same proportion that those of the first were; and that all the squares be equal among themselves.

3. If the area of the copy is to be in some precise proportion to the area of the original, the area of the parallelograms, and consequently of the squares, must be accurately in that proportion; and therefore the sides of the squares must be as the *square roots* of the terms expressing that ratio or proportion.

Thus, for example, suppose the area of the copy was required to be precisely two-thirds of that of the original; if the side of the squares of the first parallelogram be taken as unity, the side of the squares of the other for the copy must be $\sqrt{\frac{2}{3}}$ or $\frac{1{\cdot}4142}{1{\cdot}7320} =$ $\cdot 8164$, since the area of a square, having its side equal to this decimal, will be equal to two-thirds of that having one for its side.

By Problem 28, the side of the square for the copy may be obtained in any proportion from $\frac{1}{10}$ to 10 times the original; it will be advisable to take a square, containing 4 or 9, or more of the small squares of the first parallelogram, and to make the necessary constructions on it; the resulting line must be divided into two or three parts for the side of the square required.

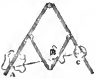
When a picture or drawing is to be copied, on the surface of which no lines can be allowed to be drawn for fear of injury, artists and engravers stretch silk threads across, so as to form a net work of squares, held by common pins on the edge of the picture, taken out of its frame, or else on a wooden frame, made expressly to fit into the former. The sides of the parallelogram are carefully drawn on this frame, and divided by the spring dividers in the manner above described, the pins being fixed in the points of division, and the threads passing all on the same side of them, form the squares of precisely the same size. It may be easily imagined with what care this must be done, when the subject is a portrait by Vandyck to be reduced to one-ninth of the original, and where consequently a trifling error in the size of one row of squares would cause a distortion in the copy fatal to it.

Some draughtsmen, who have often occasion to make use of this mode of reducing original drawings, have a light wooden or brass frame, with fine holes drilled round the four sides, through which the threads being passed, a permanent frame of squares is formed, which can be laid on any drawing without injury to it.

It is sometimes required to copy certain figures, as plans, &c.

in different proportions in two directions, for the purpose of bringing them within a rectangular figure, not similar to that in which the originals are included; this is effected by dividing the sides of the rectangle of the copy, as before, into the same number of divisions into which those of the original are divided; but the lines drawn through these unequal divisions will form rectangles not squares; and in these the corresponding parts of the original are to be copied, taking care to preserve the same unequal proportion between the detailed parts included in each space.

Artists being perpetually under the necessity of copying drawings or figures to a different scale, many instruments, called Pentagraphs, have been invented to facilitate the operation.

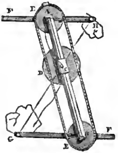


The common PENTAGRAPH, of which the annexed figure represents an improved form, consists of four brass rules, two about 15 or 18 inches, and the other two about half that length, put together with joints so that they form a parallelogram; the weight C slides on, and can be clamped at any part of one of the longer arms; a point on the under side pierces into the drawing board to fix the instrument, C being the centre on which the whole moves: the blunt *tracer* is at A, on the other long arm, and a socket which carries the pencil slides on one of the shorter at B. To make the adjustment, the tracer A, the pencil B, and the centre of motion C, must be placed in a line, so that A C shall be to B C in the proportion in which the original is to be reduced: when set, from the construction of the frame, C, A, and B will always be in a right line, and C A will preserve the same constant proportion to C B, so that whatever motion may be given to A the pencil B will move over a *similar* line. If the original figure were to be enlarged, the tracer must be placed at B and the pencil at A. (Note Q.)

There are small rollers under the joints which carry the instrument on the paper, to diminish the friction.

Another instrument for the same purpose has been invented by Professor Wallace of Edinburgh, called the EIDOGRAPH; it is more elaborate and expensive, but far superior to the Pentagraph in the precision and beauty of its application.

A B is a triangular mahogany bar, having a longitudinal groove in it; an arbor, fixed to the weight or fulcrum D, on which the socket C turns, passes through it; by this means the bar can slide in the socket: at each end of the bar and on its under side is a wheel E, each of precisely the same diameter and turning on vertical axes, in pipes fixed to the bar; on the underside of each wheel is a socket in which the arms F slide; one of these has a tracer and the other a pencil at opposite ends, G and H; a steel watch-chain passes round the wheels, and is clamped to them to prevent its slipping. To adjust the instrument, the bar A B is slid in the socket C, and the arms F in their sockets, so that B C and B G shall have the same proportion to A C, A H that the original has to the intended copy. These adjustments are made, by means of graduated scales, on the bar and arms for the purpose; by this the tracer and pencil are brought into a line with the centre C. (Note Q.)



The motion of the tracer is communicated to the pencil by the chain round the wheels, and a string passing over a pulley and round pins at A and C is brought to the artist's hand to enable him to raise the pencil where the outline is discontinued.

The action of this instrument in practice is so correct, that the outline of a figure can be reduced by means of it to an almost microscopic minuteness, with an accuracy which defies the most critical examination to detect an error: this great superiority chiefly arises from the small degree of friction occasioned by the mode in which the motion is communicated to the pencil; in the Eidograph, the friction of the axes of the wheels and of the centre of motion at C is alone to be overcome; while in the Pentagraph there are five centres of motion, and the friction of the rollers on the paper in addition: this advantage enables the artist to move the tracer of the former over the complicated outline of a countenance with a precision and spirit that is impracticable with other machines.

Problem 35.

To construct regular polygons.

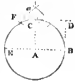
Case 1. If the polygon is to be inscribed in a given circle, the circumference must be divided into the requisite number of equal arcs, either by trial or by the methods explained in Problem 36; and these points of division being joined by chords, the figure will be completed. (Geo. III. Props. 26, 27, 28.)

Case 2. If the polygon is to be constructed on a given right line A B, as one side.

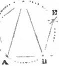
1. The *equilateral triangle* requires no observation. (Geo. I. Prop. 42.)

2. To construct a *square*. Draw a perpendicular at either end of A B (Problem 1.), and set off on it the length of A B; then with this line for a radius, arcs described from the extremities of these two vertical lines will intersect in the fourth angle.

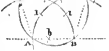
A distance A B being given, two other points terminating the sides of a square on A B can be found by the compasses alone. On A or B, with A B for a radius, describe a circle; and from B set off the radius three times to E; with the chord B F, from B and E describe arcs, cutting in *a*, and with A *a* from B and E describe arcs cutting the circumference in C: from B and C with A B describe arcs cutting in D, and A, B, C, D will be the angles of a square. (Note R.)



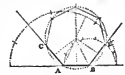
3. To construct a *pentagon*. On A B for a base, construct an isosceles triangle A B C, having the angles at the base double those at the vertex C (Problem 27.); from A, B and C, with A B for a radius, describe arcs intersecting in D and E; then A D, D C, B E, C E being drawn, the figure will be completed; if the centre of the circle circumscribing the triangle A B C be found, the points D and E will be more accurately determined or verified.



A distance A B being given, the other angular points of a pentagon on A B can be found by the compasses alone. On A and B, with A B for a radius, describe two circles, and divide A B medially in *b* by the construction in Problem 6; from A and B set off the distance B *b*, or the greater segment, three times on each circle to G and H; then from these two points, with A B as a radius, describe arcs cutting in I; A B G H I will be the points of a pentagon. (Geo. III. Prop. 28.)



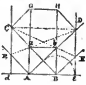
4. To construct a *hexagon*. Describe an equilateral triangle on A B, and with the side for a radius from the vertex describe a circle, the radius of which, A B, will divide it into six parts for the angular points: this requires the compasses alone.



5. To construct a *heptagon*. On A and B, with any radius, describe two semicircles, and produce A B for a common diameter; divide each semi-circumference into seven equal parts by trial, and draw A C, B D, through the second points of division from the diameter; make A C, B D each equal to A B, and bisect all three lines by perpendiculars, which will meet in the centre P of the circumscribing circle; A B will divide the arc C D of this into four equal parts, if the construction have been correctly made; if the angles C A B, D B A be bisected by means of the arcs of the semicircles, the bisecting lines will also pass through the centre P, and will verify the construction.

The principle of this construction is the only one by which a regular polygon of a prime number of sides, as 7, 11, 13, &c. can be *geometrically* made on a given line for a side: the semicircles on A B produced, must be divided, by trial, into as many equal parts as the polygon required has sides; and the second point of division from the diameter must in every case be taken;* by this means the angle of the polygon can be formed at each extremity of A B: the centre of the circumscribing circle may then be found by bisecting the angles, or the three containing sides.

If lines be drawn from A and B through the other divisions of the semicircles, they will pass through the angular points of the polygon, and may thus serve to verify the constructions.



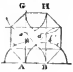
6. To construct an *octagon*. On A B describe a square A B, *a b*, draw and produce the diagonals A D, B C, and parallel to them draw B F, A E; make A E, *a* C, B F, *b* D equal to A B; draw C E, D F parallel to A *a*, or at right angles to A B; then from C and D draw lines C G, D H parallel to A D, B C, and make them also equal to A B; and the remaining side G H must be drawn parallel to A B.

This construction admits of many variations and verifications: thus if A *d*, B *e* in A B, produced each way, be made equal to half the diagonal A *b*; and the lines *d* C, *e* D, drawn perpendicular to *d e*, and if a square be described on *d e* the sides C E, D F, G H will be in those of the square, and may be found by setting *d* A from each angle along them. Again, the lines G F, E D, if the construction is correct, should pass through *b* and *a*.

---

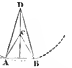
* The exterior angle of a polygon of 2 *n* sides is $\frac{4 \times 90}{2\,n} = \frac{2 \times 180}{2\,n} = \frac{180}{n}$ (Geo. I. Prop. 20.) that is, equal to the arc of one division of the semicircle, divided into *n* parts, or two divisions if it be divided into 2 *n* parts.

From A and B, with the radius AB, describe semicircles, and set the radius twice off on either circumference; with the chord of these two divisions, from each extremity of the diameter, A B produced, describe arcs intersecting in *a*, *a*; then *a a* being drawn parallel to A B, it will be the side of the square on G H, the opposite side of the polygon; and if A *a* be set off from *a′a′*, the side of the square on A B, to G and H, the side G H may be thus obtained.



7. To construct a *decagon*. On A B for a base, construct an isosceles triangle A B C, having its angles at the base double those at the vertex (Prob. 27.); then C will be the centre of the circumscribing circle, the circumference of which will be divided by A B into ten equal parts for the angular points. (Geo. III. Prop. 28.)

8. To construct a *dodecagon*. On A B describe an equilateral triangle A B C; bisect A B by a perpendicular through C, and make C D in it equal to A C or A B; then D will be the centre of the circumscribing circle. Or the angles A B D, B A D may be made equal to 75°, or $\frac{5}{6}$ of a right angle, which will give the point D at once.



Every mode of constructing a polygon, on a given line for a side, is practically liable to error, since the object of the proceeding is to obtain a large magnitude deduced from a small one; in every case, therefore, the centre of the circumscribing circle should be obtained, and the construction verified and completed by means of the circumference.

The following Table will be found of great use in all Problems relating to regular polygons.

No. of sides.	Name of Polygon.	Angle of centre.		Angle of Polygon.		Length of *Apothem, rad. = 1	Length of side, the rad. = 1	Length of rad. the Apothem = 1	Length of the radius, the side = 1	Multiplication for areas.
		°	′	°	′					
3	Trigon. Eq. Tr.	120	0	60	0	·5000	1·7320	2·0000	·5773	0·4330
4	Tetragon. Squ.	90	0	90	0	·7071	1·4142	1·4142	·7071	1·0000
5	Pentagon	72	0	108	0	·8090	1·1755	1·2360	·8507	1·7205
6	Hexagon.....	60	0	120	0	·8660	1·0000	1·1547	1·0000	2·5980
7	Heptagon	51	$25\frac{5}{7}$	128	$34\frac{2}{7}$	·9010	·8677	1·1095	1·1524	3·6339
8	Octagon	45	0	135	0	·9239	·7654	1·0823	1·3065	4·8284
9	Enneagon....	40	0	140	0	·9397	·6840	1·0642	1·4619	6·1818
10	Decagon.....	36	0	144	0	·9511	·6180	1·0515	1·6180	7·6942
11	Hendecagon..	32	$43\frac{7}{11}$	147	$16\frac{4}{11}$	·9595	·5634	1·0422	1·7747	9·3656
12	Dodecagon ...	30	0	150	0	·9659	·5176	1·0352	1·9319	11·1961

If a circle were given, and it were required to describe a heptagon in it: ·8677 being taken from the scale, the unit of which was equal to the radius, that distance will step round the circumference seven times, and thus give the angular points of the polygon.

If a line were given on which it were required to describe a hendecagon: a scale to which the given line were the unit being found, the distance 1·7747 must be taken off from it, and an isosceles triangle constructed on the line with that distance for its sides, the vertex will be the centre of the circle circumscribing the required polygon.

In the same way the length of the apothem, or the angles at the centre and side, might be made use of to construct the elementary triangle, in certain cases, according as the circle or side were given.

If it were required to ascertain the perimeter of an enneagon, the radius of the circle being taken as unity, the number ·6840 must be multiplied by 9, which will give 6·1560 for the perimeter.

---

* The apothem of a regular polygon is the perpendicular from the centre to the side; and which of course bisects that side. (Geo. III. § 4. Def. 12.)

If ·5176, the length of the side of a dodecagon, be multiplied by 12, it will give 6·2112 for the perimeter; the half of this, or 3·1056, multiplied by ·9659, the length of the apothem, gives 2·99969910 for the area; that of the circle being 3·1415916.

If the side of an octagon were 2·61, this number multiplied by 4·8284 will give the area.

These examples are sufficient to show the use of the Table.

§ PROBLEMS RELATING TO THE CIRCLE.

Problem 36.

To divide the circumference of a given circle into any number of equal parts.

The following constructions, which require the compasses alone, are best made with spring dividers, two or three pair of which, being employed, the distances repeatedly wanted may be kept unaltered.

1. With the given radius describe the circle (Fig. 1.) and divide it into six parts, in B, C, D, E, F, G. Then B E is a diameter and divides it into two, B D is the chord of $\frac{2}{6}$ or $\frac{1}{3}$, and the circle is divided in B, D, F, into three parts.

2. From each end of the diameter B E, with the chord B D or C E, describe arcs intersecting in X,* then the distance A X being set off from B and E, the circumference will be divided into four parts in H, I.

3. The arc described with the radius A B from X, X as centres, will cut the circumference in K, L, M, and N, which points bisect the quadrants B H, H E, &c., and thus divide the circle into eight parts.

4. The radius A B set off from H, I, to O, P, Q, and R, bisects the arcs B C, D E, E F, and G B, which completes the trisection of each quadrant, and therefore divides the circle into twelve parts.

5. The radius A B, set off from K, L, M, and N, both ways from each point, will bisect the two arcs on each side of the

Fig. 1.

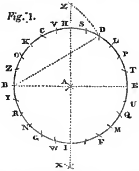
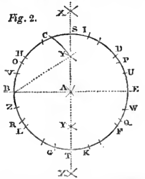


Fig. 2.



---

* In all these constructions, in order to ensure greater accuracy, the arcs should be described on both sides of the line, joining the centres: thus the point X should be found on both sides of the diameter B E.

vertices of the diameters B E, I H in S, Y, U, V, T, W, Z, &c., and thus complete the division of the circumference into twenty-four parts.

Any further sub-division, consisting in bisection of the arcs already found, is best performed in practice by trial: thus each arc of the twenty-fourth part being bisected will divide the circle into forty-eight parts.

All the foregoing constructions, by which the circumference is divided into twenty-four parts, are performed, it will be seen, by three distances only, the radius A B, the chord B D, and A X—consequently if these be kept during the operation unaltered in the dividers, the division is performed with the greatest accuracy.

6. With the distance A X as a radius from O, P, Q, R, (Fig. 2) describe arcs intersecting in Y, then the distance B Y or E Y will divide the circumference into five parts in B, C, D, F, and G.

7. The distance A Y will bisect the arcs B C, C D, D F, &c., in H, I, E, K, L, and thus divide the circle into ten parts.

8. The distance B Y set off from S, T, the vertices of the diameter perpendicular to B E, will bisect the arcs D E, B H, E F, B L, in the points U, V, W, Z, and will thus give $\frac{1}{20}$ of the circumference, the same distance being set off from these points will bisect the other arcs of the decagon.

The division into forty parts may be effected by trial by bisecting the arcs last found.

The construction by which the circle is divided into five, ten, and twenty, is as simple as the former, being performed by three distances only, A X, B Y, and the radius. (Note R.)

9. A circle can be divided in 7, 9, 11, 13, or any number of parts expressed by a prime number greater than 5, either by trial; (Geo. III. § 7, Schol.) or by making use of a table of chords.

Problem 37.

*To describe a circle that shall have its circumference, or perimeter, equal to a given right line, or its area equal to a given figure.**

1. Find a fourth proportional to two lines in the ratio of 3·1416 : 1, or in the ratio of 355 : 113, and the given line; (Prob. 9.) and this fourth proportional will be the *diameter* of the required circle.

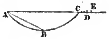
If the perimeter is given arithmetically. Divide the number by 3·1416 for the diameter, which must be taken from the scale, and the required circle described on it.

---

* *Theoretically*, no right line, or right line figure, can be constructed exactly equal to the circumference or area of a circle, but the difference in *practice* is too small to be appreciable.

2. Construct a square equal to the given figure (Prob. 29.) Then 1·1284, taken from a scale on which the side of this square is unity, will be the diameter of the circle required, or if the diagonal of the square be made unity, ·7854 will be the diameter of the circle of the same area.

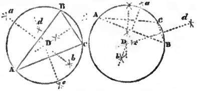
The following mode of finding the length of a given arc of a circle *nearly* is frequently of use to mechanics:—



Draw the chord A C of the given arc, bisect the arc in B, make A D equal to twice the chord A B, divide C D into three equal parts, and add one of these to the line A D, then A E so augmented will be the length of the arc *nearly*. (From Nicholson's Carpenter's New Guide.)

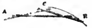
PROBLEM 38.

To describe a circle to pass through three given points A B *and* C.



Bisect the distance between the two most distant points A, B, by a perpendicular *a b*; bisect the distance between the remaining point C and either of the others by a perpendicular *c d*, (Prob. 1.): then the point D in which these perpendiculars intersect will be the centre of the circle required. (Geo. III. Prop. 55.)

If the three points are so nearly in a right line, that the two perpendiculars would only meet in an inaccessible point; or the point of concourse be too far off to admit of the circle being described by the compasses, the following method must be adopted:



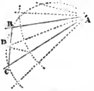
Lay two rules to touch the points in the manner shown in the figure, and pin them together where they cross at *c*; a third rule must be also fixed across them to keep them firmly at the same angle: if the rules thus fixed be moved so as always to touch the two exterior points, a pencil held in the angle at *c* will describe the arc of the circle that will pass through them.

When circumstances will admit of it, it will be best to stick two pins in the drawing board in the two exterior points, the rules can then be moved without trouble against them.

When the rules are fixed, allowance must be made for the thickness of the pencil point, so that the circle it describes shall pass through the points, this is shown by dotted lines in the figure.

By means of this problem a circle may be described round a given triangle, and if the three angles of any triangle be bisected, the intersection of the bisecting lines, which will cut one another in one point, will be the centre of a circle that may be inscribed in that triangle.

Three points being given, other points which will lie in the circumference of the circle that passes through the given ones, can be found by the following construction:—Let A, B, C, be the three given points: from either point, as A, draw lines through the other two; and from A as a centre with any radius, describe an arc of a circle cutting the lines A B, A C; bisect the included arc, and step the distance along the circumference, each way, and draw radii from A through the points. Join B C and bisect it by a perpendicular which will cut the line bisecting the angle B A C in a fourth point D of the circle. With the distance B D or C D, from B or C, cut the adjacent radius from A in other points, which will also be in the same circumference; then with the distance B C, from B or C, cut the next radius but one each way, and these will be new points, through all of which the circle will pass that would pass A, B, and C. (Note S.)

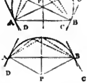


If the same construction were applied to either of the other given points, another set of equidistant points in the same circle would be obtained if required.

Problem 39.

Through two given points A, B *to describe an arc of a circle that shall contain a given angle.*

Draw a line A B joining the given points, at either, or both points, construct an angle A B C, or B A D equal to that required to be contained by the arc, but on the contrary side of A B to that on which the arc is to lie; draw, A P, B P, perpendicular to A D, B C, the lines forming the angles; and P, in which the perpendiculars meet, will be the centre of the arc.

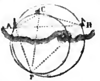


But if the angle was made at one point only, then the line A B, or chord joining the two given points, must be bisected by a perpendicular which will intersect the perpendicular from A or B, in P the required centre.

It is more accurate to construct the angle at both ends of the line, and it is even necessary, when the angle is very large, also to draw the perpendicular bisecting the chord, in order to verify the intersection of the other lines.

An important and useful application of this problem in coast surveying will be most easily explained by an example.

Suppose the angles subtended by any three objects, A, B, C, from any station be taken, the real distance from each other of which are known, and it is required to ascertain the situation of the observer relatively to them. He constructs the triangle A B C, having its three sides proportionate to the known distances of the objects; through A and B he draws, by the construction given in this problem, an arc of a circle that shall contain the angle he observed to be subtended by A B. Through B and C he draws another arc containing the angle subtended by B, C: it is obvious that these arcs or circles will cut each other in the point P, which represents the station of the observer, and he can measure by the scale to which he drew the triangle his distance from them.



If the observer be careful, he will also construct the arc that should pass through A and C, and contain the angle subtended by them, in order to verify the point found; as this third arc ought to cut the two others in the same point P, if the angles are correctly observed.

By this means a ship sailing along the coast of a country of which she possesses the map, can at any time tell her position with regard to three land marks, the angles of which are observed, and the distances from each other of which are known from the chart, and can thus note down the soundings and bearings which are the objects of a coast survey. From the superiority of the instrument used, the sextant, this mode of determining a situation is more accurate than by using bearings.

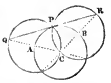
If it should happen that the observer is in the circumference of the circle passing through the three points at the time of observing the angles, the position cannot be determined by this means.

PROBLEM 40.

To draw a tangent to a circle through a given point P.

Case I. If the point P be in the circumference:

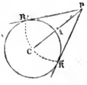
From P set off the radius P C of the given circle each way on the circumference to A and B, and from these two points



with the same radius, describe arcs of circles, which will obviously pass through P and C; draw A C, B C cutting the arcs in Q and R; then R Q being drawn it will be the tangent to the circle at P as required.

It will be seen that the lines C Q, C R will pass through the points in which the arcs cut the circle again, this will be an additional verification.

This is a more accurate construction than that of drawing R Q at right angles to the radius C P in the ordinary way, and is quite as expeditious.



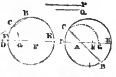
Case 2. If the given point P be not in the circumference:

Join P, and C the centre of the circle, bisect P C in A, and from A with A P or A C intersect the circumference of the circle in R, then a line drawn through P and R will be the tangent required. (Geo. III., Pr. 56.)

In the latter case, if the tangential point is not required to be very accurately determined, the tangent may be drawn by a rule through P touching the circle.

PROBLEM 41.

To draw a right line through a given point A, *so that it shall be cut by the circumference of a given circle in a given ratio* P : Q.

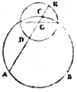


Through A draw a diameter D E, and make A F in it a fourth proportional to P, Q, and A D, (Prob. 9.) then make A G a mean proportional between A F and A E: from A with A G, for a radius cut the circle in C and a line drawn through A and C will be cut in C and B as required, that is A B : A C : : P : Q. (Leslie, An. I. Prop. 9.)

If P and Q were the sides of a rectangle, by this construction a rectangle similar to that given one may be made; this, though too circuitous a mode for common application, may be occasionally useful.

Problem 42.

Through two given points A, B *to describe a circle which shall divide the circumference of another given circle in two equal parts.*

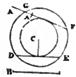


Through either given point A, draw A E through G, the centre of the given circle, make G C a third proportional to A G and D G; then if through A, B and C, a circle be drawn, (Prob. 38.) it will cut the original circle in the ends of a diameter. (Leslie, An.)

This construction is used in the stereographic projection of the sphere, in constructing maps.

Problem 43.

Through a given point A, *to draw a line so that the segment intercepted by the circumference of a given circle, shall be equal to a given line* B.

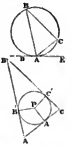


With B for a radius, from any point D in the circumference of the given circle cut this in E, so that D E being joined, D E shall be equal to B; from C the centre describe a circle to touch D E; then a line A F drawn through A, a tangent to this last circle, will have the segment G F equal to B. (Geo. III., Prop. 4, Cor.)

Problem 44.

In, or about, a given circle to construct a triangle similar to a given one.

1. Through any point A in the given circle, draw a tangent D E, and make the angles D A B, E A C equal to two of those of the given triangle; join B, C the points in which the legs A B, A C cut the circle; then A B C will be similar to the given triangle and will be inscribed in the given circle. (Geo. III. Prop. 62.)

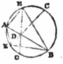


2. From the centre P of the circle draw P B', P A', P C', making the angles A' P B', B' P C' equal to two exterior angles of the given triangle; draw A B, A C, B C, perpendicular to P B', P A', P C' respectively, or make those lines tangents to the circle, at B', A' and C', then A B C will be a triangle similar to the given one and about the given circle. (Geo. III. Prop. 62.)

Problem 45.

From two given points A *and* B *in the circumference of a given circle, to draw two lines to meet in a third point in it, which shall be to each other in a given ratio.*

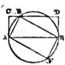
Bisect the arc A B in the point C, join A B and divide it in D in the given ratio (Prob. 12): draw a line through C and D cutting the circle in E, the point required, join A E, B E, and they will be in the given ratio. (Geo. II. Prop. 50, and III. Prop. 14. Cor. 2.)



Problem 46.

In a given circle to describe a rectangular parallelogram equal to a given rectilinear figure.

On A B, the diameter of the given circle, describe a rectangular parallelogram A B C D equal to the given figure (Prob. 29.) from the point E where the side parallel to A B cuts the circle, draw lines to A and B each extremity of the diameter, and which will obviously contain a right angle. Complete the parallelogram by lines from A and B parallel to A E, B E which will also intersect in the circumference in F: then A E B F is the parallelogram required.



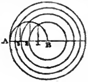
If the area of the given figure is greater than the square on the radius, the problem is impossible.

Problem 47.

To divide the area of a given circle into any number of equal parts.

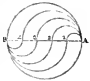
1. By concentric circles.

On A B, any radius of the given circle, describe a semicircle and divide the same radius A B into the required number of equal parts, as in 1, 2, 3; from the points of division draw perpendiculars to cut the semicircle in points, through which concentric circles to the given one being described, they will divide it into the required number of equal parts.



2. By semicircles described on the diameter.

Divide the diameter into the required number of equal parts, for example five, in the points 1, 2, 3, 4. On A 1, A 2, A 3, A 4 as diameters, describe semicircles on one side of the diameter A B. Then on B 4, B 3, B 2, B 1 as diameters, describe semicircles on the contrary side of A B. These semicircles thus combined two by two will divide the area of the circle into five equal spaces, each included by equal circumferences. (Note T.)



A circle can only be divided into equal parts by straight lines, *geometrically*, by approximation, the area of a circle, or of a segment, being incommensurable with a right line figure; and the mode of doing it approximately would be too complicated to be of any *practical* utility. The following properties of the areas and circumferences of the circle of segments and of those figures called *lunes* may be occasionally found of service.

1. The *similar* segments described on the base and perpendicular of a right angled triangle are together equal in area to that on the hypotenuse.

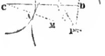
2. If on the hypotenuse of a right angled triangle a semicircle be described to pass through the right angle and semicircles be described also on the base and perpendicular, the area of the triangle will be equal to that of the two lunes thus formed, added together.

3. If from the extremities of any side of a square, circles be drawn one with the side for a radius, and the other with the diagonal for a radius, the area of the smaller of the lunes thus formed will be equal to that of the square.

PROBLEM 48.

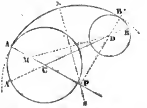
Given two circles, not concentric, to draw a third to touch them both, or so that the tangents at the points of contact shall be common to it and the given circles.

Let C and D be the centres of the given circles: from any point A in the circumference of either of them draw A P a diameter; take A M in it equal to the radius of the other circle: bisect M D by a perpendicular produced to meet A P in P. Then a circle from P, with P A for a radius will touch the given circles in A and B; and as the point A



may be assumed, the circle may be made to touch one of the given circles in a given point as A.

If the centres C, D of the given circles be joined by a line cutting them both in A′ and B′, then a semicircle on A′ B′ will fulfil the condition.



From the comparative difficulty of drawing all curves except the circle, recourse is had on many occasions to an imitation of the required curve composed of segments of circles of different radii; that these may form a continued line without any angles, it is necessary that the centres of two contiguous segments should be in a line with the point of junction; to effect this, the construction in this problem is required.

The following table of factors to facilitate geometrical constructions relating to the circle will be found of use.

Let *c* be the circumference of a circle, the diameter of which is unity, and let *x* be the side of a square equal in area to it, and let *a* be the area of the circle.

$$c = 3{\cdot}14159 \text{ \&c.}$$

$$2c = 6{\cdot}28319 \text{ \&c.}$$

$$\frac{c}{2} = 1{\cdot}57080 \text{ \&c.}$$

$$\frac{c}{12} = 0{\cdot}26180 \text{ \&c.}$$

$$\frac{c}{360} = 0{\cdot}00873 \text{ \&c}$$

$$\frac{1}{c} = 0{\cdot}31831 \text{ \&c.}$$

$$\frac{2}{c} = 0{\cdot}63662 \text{ \&c.}$$

$$\frac{360}{c} = 114{\cdot}59156 \text{ \&c.}$$

$$x = 0{\cdot}88623 \text{ \&c.}$$

$$a = 0{\cdot}78540 \text{ \&c.}$$

§ CONIC SECTIONS.

NEXT in importance to the circle are the *Conic Sections*, so called from their being the curves produced by cutting a *cone* by a plane; the demonstrations of the manner in which they are derived from the cone, and of their principal properties, will be found in the Appendix to the Treatise on Geometry. The Problems relating to them given in this work are those most useful to the artist and mechanic; and several of them are intended to enable the draughtsman to dispense with the instruments invented for describing the ellipse, the most important of these curves, as they are necessarily expensive and complicated. In pursuance, however, with the plan hitherto adopted, a general description of the two principal elliptographs, as they are termed, is given.

As many of the terms relating to these and other curves may not be familiar to the beginner, and as they necessarily occur frequently in the following problems, a brief explanation of some of them here may prevent the necessity of recurring to other works.

1. The *axis* of a curve is a line which divides it into two equal and similar parts; in the *ellipse* and *hyperbola*, two of the conic sections which have two axes terminated by the curves, the longer one is termed the *transverse*, or *major*, and the other the *conjugate*, or *minor axis*. In these curves, these axes are always at right angles, and mutually bisect each other.

2. In the ellipse and hyperbola, the *centre* is a point through which all lines drawn and terminated by the curve, will be bisected in that point, and all such lines are called *diameters*. The axes of these curves are, therefore, also diameters, since they pass through the centre.

In the parabola, a curve which has no centre, the diameters are lines parallel to the axis, and terminated at one extremity only by the curve.

3. The extremities of all diameters and of the axes are called *vertices*: the diameters of the parabola have only one *vertex*.

4. A *double ordinate* is a line bisected by a diameter, and terminated by the curve, and is always parallel to the tangent to the curve at the vertex of the diameter: half of the double ordinate is an ordinate; and in the ellipse and hyperbola, the diameter to which the ordinates and tangent are parallel is said to be *conjugate* to the former.

5. The third proportional to any two conjugate diameters of an ellipse or hyperbola is called the *parameter* of those diameters;

a double ordinate passing through the focus is called the *parameter* of a parabola.

6. The *foci* of the ellipse and hyperbola are two particular points equally distant from the *centre*, one on each side of it, and lying in the *transverse axis*. For their particular properties, see Geo. App.

In the ellipse and hyperbola, the distance of either focus from the centre is called the *excentricity*.

The parabola has only one *focus*.

7. The *directrices* are certain lines at right angles to the transverse axis produced; and lying out of the curve at equal distances from the centre. (Geo. App.)

The parabola has only one *directrix*.

8. The *asymptote* to a curve is a line which constantly approaches nearer and nearer to the curve, but which can never touch it, however far it may be continued.

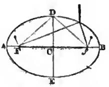
9. A *normal* to a curve at any point, is the right line perpendicular to the tangent to the curve at that point: thus the radii of a circle, and the axis of any curve are normals to the curve.

10. An *abscissa* is the segment of a diameter cut off from the vertex by an ordinate.

The curves called the conic sections may be easily produced by means of a thin ring of any kind; if this be held *parallel* to a wall, between it and the flame of a candle or lamp, the shadow of the ring will be a *circle;* if the ring be held obliquely to the wall, its shadow will be an *ellipse;* if the ring be held *horizontally* above the flame, or so as to surround the candle, its shadow on the upright wall will be an *hyperbola;* and lastly, if it be still held *horizontally* touching the wall, and the candle be placed exactly under the point opposite to that in which it touches, the shadow on the wall will be a *parabola.*

In all these cases, the rays of light from the flame which fall on the ring constitute the cone of which these shadows are the sections; but it must be remarked that the true curves would only be produced on the supposition of the ring being a perfect circle, with no substance, or thickness; and the flame of the candle a luminous point: it being impossible, of course, to fulfil these conditions, the shadows produced will only be approximations to the real conic sections.

§ PROBLEMS RELATING TO THE ELLIPSE.



Let A B, D E, two lines at right angles to, and mutually bisecting each other in the point C, be given as the axes of an ellipse.

The semi-transverse A C, or B C being taken for a radius, and arcs described from D, E, the vertices of the conjugate, as centres; these arcs will intersect each other in the points F, *f*, in the transverse axis which will be the *foci*. (Geo. App., Pr. 21.)

If two pins, or still better two *needles*, be fixed in the board in the points F and *f*, and the ends of a thread passed round them, tied in a knot at D or E, then a pencil being put within the thread, and this kept equally stretched, while the pencil moves round, an ellipse will be described, which will pass through the four points A, B, D, and E.

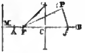
The pins should not be fixed upright in the board, or else the thread will slip up them, they should incline a little outwards from each other in the direction of the transverse axis. Allowance must be made in knotting the thread for the thickness of the pencil, otherwise the ellipse will not pass through the points meant to be the vertices, but within them. It is advisable to cut the pencil to a fine point, and to cut a small notch in the wood, for the thread to lie in, as near the point as possible, this will prevent it from slipping up or off the pencil, during the motion of it.

The thread had better be silk, as less elastic, and therefore not stretching by the tension used to keep it tight; but with all these precautions it is impossible to draw a small ellipse, or one whose mean diameter is less than three inches with anything like precision by this means, and, therefore, to the artist it is nearly useless.

The more nearly D E is equal to A B, the nearer do the foci F *f*, approach the centre C, and the *rounder* the ellipse becomes; and when D E is equal to A B the foci coincide with C the centre, and the ellipse becomes a circle; which may be considered as an ellipse with equal axes.

As D E becomes shorter in proportion to A B, the foci are more distant from the centre C, and the ellipse becomes more flat, or, as it is termed, more *excentric*, and if D E were supposed reduced to nothing, the foci F *f* would be in A and B, and the ellipse would become a straight line. The circle and straight line may, therefore, be considered as the limits of the ellipse.

As any number of concentric circles may have a centre common to them all; so any number of concentric ellipses may have their foci and centre in common, and as the radius, or a point in the circumference, is necessary to limit the individual circle, the centre of which is given, so some analogous data are required for the ellipse; as the vertex of either axis, or a point in the curve; or the directrix, if the centre and foci are given; since no two ellipses can have a centre, the foci and either of the other data in common.

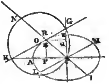


1. If the indefinite axes, the foci F *f*, and the point M where the directrix cuts the transverse axis, are given, the vertices of this transverse axis are found by making C A, C B a mean proportional between C F and C M, (Prob. 7.) then A B is the axis, (Geo. App., Pr. 21, Cor. 2.) and arcs described from F and *f* as centres with A C or A B for a radius, will intersect in the vertices of the conjugate axis.

2. If the point P in the curve, and the foci F *f* are given, draw a line through the foci and bisect F *f* in C, which will be the centre; draw lines from F and *f* to P, then half the sum of F P, *f* P, will be the semi-transverse A B or B C, and the conjugate axis may be found as before.

PROBLEM 49.

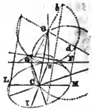
Two conjugate diameters L M, I H *being given by two lines; to find the axis of the ellipse, the curve itself not being given.*



1. Produce C H to N, so that H N shall be a third proportional to C H, C M, (Prob. 8.) and bisect C N in R; through H draw a line parallel to L M, and draw R O perpendicular to C N to meet this line in O; from O as a centre with O C as a radius, describe a circle cutting the parallel through H, in G and K; and draw C G, C K indefinitely produced for the axes.

Draw H P, H Q parallel to C G, C K, and make C E a mean proportional (Prob. 7) between C Q and C G, and make C A a mean between C P and C K, then C E, C A are the semi-axes. (Lesl. G. C. L. B. I., Prop. 30.)

If the perpendicular R O intersect G K too obliquely to allow of the point O being accurately determined, it may be necessary to repeat the construction with the other diameter C L.



2. Through H draw a line parallel to L M, draw H O perpendicular to it, and equal to C L or C M; with H O for a radius, from O as a centre, describe a circle, bisect O C by a perpendicular cutting the parallel through H in F, from F as a centre with F O for a radius describe a semi-circle meeting H F in G and I,* then lines drawn from G and I through C will be the indefinite axes.

Join O G, O I cutting the circle described from O in points *b*, *d*, through which lines, parallel to O C being drawn, they will cut the axes in the vertices. (Note F.)

If L M, H I were given as two conjugate diameters, and any line C G as an indefinite diameter, a conjugate to it may be found by this construction.

For draw the parallel through H, and H O perpendicular to it equal to C M as before, let the given indefinite diameter meet H G in G, join G O, and draw O I perpendicular to it, cutting H G in I, then I C being drawn will be the indefinite conjugate sought, and the vertices may be found as before.

These constructions are of great use and frequent occurrence in linear perspective, where two conjugate diameters of an ellipse are frequently more easily determined than other data from which to describe the curve.

As all diameters bisect each other in the centre, (Geo. App., Prop. 17.) if any two lines bisecting each other are given for two diameters not conjugate to each other, the lines joining their extremities will be parallel chords, consequently lines through the centre parallel to these, and bisecting the others alternately, will be *indefinite* conjugate diameters. (Geo. App., Prop. 19, Schol.)

PROBLEM 50.

Given the axes of an ellipse, A B, D E *to determine the points where the curve would cut a given line* H G, *parallel to either of them.*

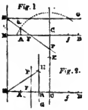
1. Let H G be parallel to A B. (*Fig.* 1.)

From D and E with the semi-axis A C for a radius, intersect the axis A B in F and *f* which will be the foci. (Les. G. C., Pr. 1, Pr. 6.) Make C M in the transverse axis produced, a third proportional to C F and C A, and through M draw a line perpendicular to A B: let the given line meet this perpendicular in H, draw

---

* The point I is out of the figure, but it will be seen that the line through C tends towards it.

Google



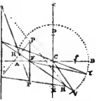
H F cutting the conjugate axis in P, and from A draw a perpendicular to A B or a parallel to the conjugate, to meet H P in Q. Then on P as a centre with the radius P Q, describe a circle which will cut, or touch, the given line in the points, or point, where the curve of the ellipse will cut it, if it cut it at all, for it is obvious that if the distance of the given line from the transverse axis were greater than the semi-conjugate, the curve would not touch the line at all.

2. Let H G be perpendicular to A B, or parallel to D E (*fig.* 2). Find the foci F *f*, and the directrix as before, and draw an ordinate through F perpendicular to A B; make the semi-ordinate, F P, a fourth proportional to C M, C A, F M. (Prob. 9.) Draw M P cutting H G in H, and from F as a centre with a radius equal to R H, describe arcs cutting H G in the points sought. (Les. G. C. B. I., Pr. 2. and 3.)

These constructions can be made with either focus, taking care to take the point M in the transverse axis produced, at that extremity next the focus employed; by making the constructions with both foci at the same time, the process is verified, and the required points more correctly determined.

Problem 51.

Given the axes A B, D E *of an ellipse, to find the points where the curve would cut a given straight line not parallel to either.*



Case 1. If the given line G F pass through a focus F, make C M a third proportional to C F and C A, and draw the directrix through M perpendicular to A B, (Geo. App. Prop. 21, Cor 3.) and let the given line passing through the focus F cut this directrix in G.

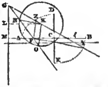
Through F draw the ordinate parallel to L M, or perpendicular to the transverse axis, and make F P a fourth proportional to C M, C A and F M, as in the preceding Prob. From G set off on the directrix G L, both ways equal to G F, then draw lines through both points L and P, and they will cut the given line in H where the curve would cut it. (Les. G. C. L., Pr. 4.)

Case 2. If the given line G C pass through the centre C, from the point G, in which the given line meets the directrix, draw G F through the focus F. On A B, the transverse axis, as a diameter, describe a circle, or mark the points T and V where this circle would cut G F; draw C T, C V from the centre C,* then, parallel to C T, C V, draw lines through F, meeting the given line in Y and Z, the points in which the curve would cut G C.

By this second case any diameter to the ellipse is found, and by Prob. 55, the conjugate diameter may be determined.

Case 3. If the given line do not pass through the focus or centre.

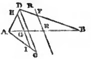
Take any point K at pleasure in the given line. Find (Prob. 9) a fourth proportional to C M, C A, K L, or to C A, C F, K L, the perpendicular distance of K, from the directrix, and with this fourth proportional for a radius, from K as a centre, describe a circle, cutting the line from G through F, in N and O, draw the radii K N, K O. Then lines parallel to K N, K O drawn through the focus F will cut the given line in the points Y, Z sought.



The farther from the directrix the point K is assumed, by so much the larger will the circle be, and so much the more accurate the construction.

Problem 52.

A line A B *being given as a diameter of an ellipse, and another line* C D, *bisected by* A B *at any angle, as a double ordinate; to find the parameter to that diameter.* (Def. 5, p. 76.)



Join A C, A D and B C, B D, and bisect A B in E, which will be the centre of the ellipse; through E draw a line parallel to C D cutting B D in F. This line will be the indefinite conjugate diameter. (Geo. App. Sch. Prop. 19.)

Make A G equal to E F; and through G draw a line parallel to C D cutting A C, A D in H and I. H I will be the parameter sought. (Note A A.)

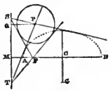
---

* The points T Z, and the lines C T, F Z are not shown in the figures, to avoid confusion.

PROBLEM 53.

Given the axes A C, C G, *the foci* F, *f, and the directrix* M S*: to draw from a given point* P, *lines which would be tangents to the curve.*

Find a fourth proportional to C M, C A and P Q, the perpendicular distance of P from the directrix; from P as a centre, with this fourth proportional for a radius, describe a circle, and from F draw F S, F T tangents to this circle (Prob. 40.), produced to meet the directrix in S and T. Then lines drawn from S and T through P would be tangents to the curve, if it were described. If the point P lie so near the directrix as to render the construction liable to inaccuracy, the other directrix at the opposite side of C must be employed.



If the point P did not lie without the curve, the problem is of course impossible; whether this be the case, may be ascertained at once by drawing lines from P to the two foci: if the sum of these be *less* than the transverse axis, or less than twice C A, the point P is within the curve; but if the sum of these lines be *greater* than twice C A, the point is without the curve: if the sum of the lines drawn from P to the foci be exactly equal to the transverse axis, then P is in the circumference of the ellipse (Geo. App. Schol. Prop. 21.), and the tangent may be drawn by bisecting the angle formed by the lines drawn to the foci from P, and drawing a line through P at right angles to the bisecting line, this perpendicular will be the tangent sought.

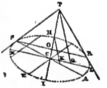
If the above construction were attempted to be applied in the supposition of the point P being in the circumference, the fourth proportional would be exactly the distance of P from the focus F. The circle on P would therefore pass through F, and consequently no tangents could be drawn to the circle from that point.

PROBLEM 54.

Given two conjugate diameters A S, C E, *to draw, from any given point* P, *two lines which would be tangents to the curve.*

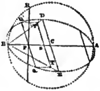
Through P draw P I through the centre C, and find, by Prob. 55, a conjugate diameter, L M, to C P and the vertices of both, as L, M, I, H. In L M make L Q equal to C H; join P Q and draw H K parallel to it. Make C O equal to C K, and through O draw S R parallel to L M, and by Prob. 55. find the terminations R S of this ordinate. Then P R, P S being drawn they will be the tangents sought. (Note C C.)

This problem, like a preceding one, is of practical utility in linear perspective, but is rather too complicated to be employed except in very particular cases.



Problem 55.

Given a line A B *as a diameter of an ellipse; and another line* P Q *as an ordinate to* A B *: to find the conjugate diameter, the curve not being given.*



Bisect A B in C for the centre of the ellipse, and through C draw a line, D E, parallel to P Q, for the indefinite conjugate (Geo. App. Sch. to Prop. 19.); draw P R perpendicular to A B at P. On A B as a diameter describe a semicircle, cutting P R in R, and make B S equal to P R. Draw S T parallel to and equal to P Q; then a line through B and T will cut the conjugate in its vertices D, E. (Note B B.)

If two conjugate diameters A B, D E were given; by the reverse of this construction the length of an ordinate at any point P might be obtained.

For draw the perpendicular P R to cut the semicircle on A B in R, and set off the length of P R from B to S. Join B D, and through S draw S T parallel to C D cutting B D in T, then S T is the length of the ordinate through P as required.

Problem 56.

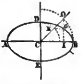
To find any number of points which would be in the circumference of an ellipse.

Case 1. Given the two axes A B, D E to find points in the curve. (See Fig. p. 78.)

1. With the semi-transverse axis A C or B C describe arcs from D and E as centres, cutting each other in F, *f* in A B which will be the foci. (Geo. Ap. Prob. 21.)

Assume at pleasure any number of points in A B, and with the two segments into which each point divides A B as radii, describe arcs from F, *f*, as centres, which arcs will cut each other in points in the curve. (Geo. App. Prop. 21. Cor. 4.)

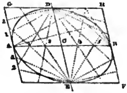
Each pair of segments will furnish four points, two on each side of the transverse and two on each side of the conjugate, and consequently, for ordinary purposes, two pair of such segments will give a sufficient number of points, together with the four vertices of the axes to allow of the curve being drawn by hand.



2. In any strait line drawn through C, that is in any diameter, make C O equal to C A or C B, the semi-transverse, and in the same make C K equal to C D or C E the semi-conjugate; through O draw a line parallel to C D, and through K one parallel to A B; these parallels will cut each other in a point in the curve. (Note D D.)

If circles be described on the two axes and any number of diameters be drawn through C, this construction will be shortened: but it would only be had recourse to on particular occasions, as, for example, when the conjngate is much shorter than the transverse, or the ellipse very excentric; in which case the first construction would be liable to inaccuracy, from the foci being so near the vertices of the axis.

Case 2. Given any two conjugate diameters A B, D E.



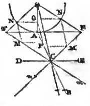
1. At both extremities of each of the diameters, draw lines parallel to the other so as to form a parallelogram F G, the sides of which will be tangents to the curve. (Geo. App. Prop. 17. Cor. 1. and 2., and Prop. 19. Schol.)

Divide each side of the parallelogram and each diameter into any, the same, number of equal parts, as in 1, 2, 4, 5, &c. From E draw lines through 1, 2, 4, 5 in A B, and from D draw lines from the points of division in A G, B H the semi-tangents parallel to D E; these lines taken two and two will cut each other in points of the curve; in the same manner the corresponding points in the other half of the curve are found; and by employing the vertices A and B and the semi-tangents D H, G D, the intermediate points are found if necessary, as will be understood from the figure. (Note E E.)

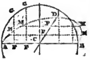
If either A B or D E, instead of being the conjugate diameter to the other, was only a double ordinate to it, in which case only one of the two would be bisected by the other; the same construction might be used for the segment of the ellipse. The two sides of the parallelogram not parallel to the ordinate would then be parallel ordinates instead of being tangents.

This construction is equally applicable to the two axes, and then the four tangents form a rectangle.

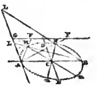
From the circumstance of the lines cutting each other at nearly right angles, the points are very clearly defined, and the facility and simplicity of the construction render this the most useful and most generally applicable of all methods of finding points in the curve.



2. Through A draw the tangent F F parallel to D E, draw A O at right angles to it, making A O equal to C D, the semi-conjugate diameter; join O C; describe a circle from O, as a centre, with O A for a radius; and draw any lines from O to meet the tangent in F. Through F and C draw a diameter: from the point N where O F cuts the circle, draw N M parallel to O C; then M, where N M cuts F C, will be a point in the curve; and if C m in F C be made equal to C M, m will be the other extremity of the diameter or another point in the curve. (Note F F.)



3. On either of the given diameters, as A B, describe a circle, and draw C O perpendicular to A B. Divide C A into equal parts in the points F, F, &c., and draw chords through the points of division, parallel to C O, or perpendicular to A B, cutting the arc A O in G, G, &c. Divide the other semi-diameter C D into the same number of equal parts in the points P, P, &c., that the semi-diameter A C was divided into; and draw lines through P, P, &c., parallel to A B; make each semi-ordinate P M equal to the corresponding chord G F, then M, M will be points in the curve. (Note G G.)



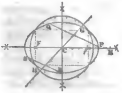
4. Draw the tangent A L parallel to D E. Make A G a third proportional (Prob. 8.) to A B and D E, and draw G F parallel to A B. From A draw any line, A F, cutting G F in F, and make A L equal to G F, draw B L cutting A F in M; then M will be a point in the curve. (Note H H.)

The same construction, made on the other side of A in A L produced or on a tangent at B, will give other points; or diameters may be drawn from M through C and C m, being made equal to C M: m, m will be points in the curve.

Problem 57.

Given an ellipse; to find the axes, foci, and directrix.

Draw any two parallel chords terminated by the curve as P Q,



R S, and bisect each of them. Then a line drawn through the bisecting points as G H, will be a diameter. (Geo. App. Prop. 17.)

The chords or double ordinates should be taken as far apart as is convenient, and should be drawn very accurately parallel, and correctly bisected, in order to obtain a true diameter.

Bisect G H thus obtained in C, which is the centre of the ellipse; (Geo. App. Prop. 17.) From C as a centre with any radius describe a circle to cut the ellipse in four points; and through these draw the two pair of parallel chords. Bisect all these, and lines drawn through the points of bisection in the opposite chords, and through the centre C, will be the axes. (Les. G. C. I. Pr. 29.)

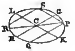
From each vertex of the conjugate axis, with the semi-transverse for a radius, describe arcs which will cut each other in two points F *f* of the transverse axis for the foci. (Geo. App. Prop. 21, Schol.)

Lastly, make C M in the transverse produced, a third proportional to C F, and the semi-transverse C A, and a line through M at right angles to C M, will be the directrix. (Geo. Ap. P. 21, App.)

The ratio of C F to the semi-transverse C A, or of this latter to C M, is what is called the determining ratio. (Les. G. C. I. Def. 3.)

PROBLEM 58.

Given an ellipse, to draw any two conjugate diameters, and to draw an ordinate to either through a given point in the curve P.



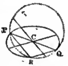
Find any diameter G H, and bisect it in C, for the centre of the ellipse, by means of two parallel chords, as in the first part of the construction of the preceding problem. Draw two chords, one on each side of, and parallel to G H, as far from it as may be convenient. Bisect each, and a line K L through the bisecting points which will likewise pass through C, will be the conjugate diameter to G H. (Geo. Ap. P. 19. and Schol.)

1. Through P draw the diameter P R, and P Q parallel to G H, and with P Q as a radius describe an arc from R cutting the curve in S; or make R S equal to P Q; or draw R S parallel to P Q. Then a line through P and the point S thus found will be the ordinate required.

2. Draw P Q, as before, parallel to G H, and bisect it, then a line through P parallel to the diameter L K, which would pass through the bisecting point, will be the ordinate to G H through P.

3. P Q being drawn through P as before, the diameter Q S will give the point S at once, but this construction is liable to be inaccurate, as a slight error in drawing Q S will materially change the point of intersection with the curve at S.

If it were required to draw two conjugate diameters in a given ellipse which should contain a given angle.



Any diameter P Q being drawn, on P Q, a segment of a circle must be described which shall contain a given angle (Prob. 39.): from the point R, where this arc cuts the ellipse, draw R P, R Q to each vertex of the diameter P Q, then two diameters parallel to R P, R Q will be conjugate to each other, and will contain the given angle. (Les. G. C. II., Prop. 1.)

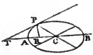
If the given angle were a right one, the required diameters would be the axes, and the segment on P Q must be a semicircle.

As the segment of the circle will cut the ellipse in two points, two pair of conjugate diameters may be found by the construction, containing the given angle.

The only conjugate diameters in an ellipse which are equal, are those parallel to the chords joining the vertices of the axes.

Problem 59.

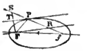
Given an ellipse, to draw a tangent to the curve through a given point P.



Case 1. If P be in the curve.

1. Draw a diameter through P, and by Prob. 58, find its conjugate, then a line through P parallel to that conjugate will be the tangent required. (Geo. Ap. P. 16. Def. 17. Prop. 19, Schol.) (Les. G. C. I., Pr. 10. Cor. 1.)

2. Draw any diameter A B, and from P draw P R, an ordinate to it (Prob. 58.); make C T in B A produced, a third proportional to C R, and C A; then P T being drawn, it will be the required tangent. (Geo. App., Pr. 18.) If A B were the transverse axis, P R must be drawn perpendicular to it.

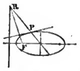


3. From P draw P F, P *f* to the two foci, and make P R in P *f* the longest of the two lines equal to P F the other; draw F R, then a line through P parallel to F R will be the tangent required. (Les. G. C. I., P. 11, 12.) (Vide note I.)

Or make P S in P *f* produced equal to P F; join F S, and bisect it in T, then a line through P T is the tangent.

Greater accuracy will be obtained by combining both these constructions.

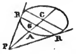
The line bisecting the angle formed by the two lines drawn from any point in the ellipse to the foci will be a *normal* to the curve at that point, or will be perpendicular to the tangent: these normals will represent the joints of the stones of an elliptic arc; the carpenter and mason, therefore, require this construction to make the moulds of the different stones.



4. Through P draw a diameter, produced to meet the directrix in R; join R and F the focus; then a line from P perpendicular to R F will be a tangent to the curve at R. (Les. G. C. I., P. 10.)

Case 2. If P be not in the curve.

1. The construction given in Prob. 54. may be used, and will be simplified from the curve being given.



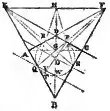
2. Through the given point, draw a diameter P C, and make C S a third proportional to C P and C A; then through S draw the ordinate S R parallel to the conjugate diameter to A C; then lines through P and R will be the tangents required.

In all cases where a point is determined by the intersection of a line with the ellipse, in problems where the curve is given, the accuracy of the construction must entirely depend on that of the curve, and whenever this has not been drawn by a machine, but by hand, it will be better not to make use of the curve, but to adopt that construction applicable to the case given in some of the previous problems where the curve is not required.

In this problem, if the curve has been described by an elliptograph, a tangent carefully drawn from the given point to touch the curve will be frequently more accurate than if the point of contact were found by the above constructions. The same has been observed of the circle, (see Prob. 42.) and the remark may be generally applied to all curves.

PROBLEM 60.

To find the axes of an ellipse that would be inscribed in a given quadrilateral, or to which the sides of this would be tangents.



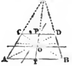
Let A B C D be any trapezium.

Produce the opposite sides to meet in the points E, F, and draw the diagonals A C, B D intersecting in G; draw E G, F G; then the four points P, Q, R and S, in which these lines cut the sides of the figure will be the tangential points, or the points where the required ellipse will touch those sides. Bisect P Q, R S, and draw E O, F O through the bisecting points, cutting each other in O; then O will be the centre of the ellipse. (Les. G. C. I. P, 17. Cor. 3.)

If one point P had been given in a side of the trapezium, as one of the tangential points, then a line P Q must have been drawn from it through G, cutting B E in Q, and B D produced to meet E F in H; then H P, H Q being drawn would have cut the other sides in R and S, the other tangential points. In this supposition, R S and P Q need not necessarily pass through E F.

The centre O of the ellipse being found; from P, one of the tangential points, draw P T through O, and make O T equal to O P; through Q, another of the tangential points, draw Q W parallel to C D, cutting P T in V, and make V W equal to V Q. Then W Q will be an ordinate to the diameter P T. (Geo. App. Def. 17. and Prop. 17.)

Through O draw a line also parallel to C D and W Q, and by Prob. 55 find the extremities of this diameter, which will be conjugate to P T, and the axes may be found from these two diameters by Prob. 49.



If two sides of the given figure were parallel, the construction is much simplified, for the other two being produced to meet in F and the diagonals drawn, a line P T through F and their intersection will be a diameter, which being bisected in O, the centre of the ellipse, a line through O parallel to A B, C D will be the indefinite conjugate diameter to P T; and an ordinate through either of the tangential points in the other sides, parallel to B D or A C, will furnish the means of determining the vertices of this conjugate as before.

If the given figure were a parallelogram, lines through the centre of the figure, which will be also the centre of the ellipse, parallel to the sides, and terminated by them, will obviously be conjugate diameters, and the axes can be found by Prob. 49, as before*.

PROBLEM 61.

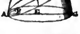
To find a rectilinear figure, or a circle, equal in area to a given ellipse.

Find a mean proportional between the transverse and the conjugate axes.

Then the decimal ·8863, taken from any scale to which this mean proportional is unity, will be the length of the side of the square equal in area to the ellipse.

And the same mean proportional between the two axes will be the diameter of a circle equal in area to the ellipse.

The following properties of the ellipse may occasionally be of service to the practical geometrician.

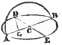


If A B be the major axis of an ellipse formed by the section of a *right* cone, of which H B and A G are the diameters of the parallel circular sections made at the vertices of A B. (See Geo. App. *passim.*)

Then the square of the transverse axis is equal to the rectangle under the diameters of the circular sections, together with the square of the slant side of the frustum; that is

$$AB^2 = AG.\ BH + AH^2.$$

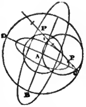
And the square of the conjugate D E is equal to the rectangle under the diameters of the same circular sections, or $DE^2 = AG \cdot HB$, and the distance between the foci is equal to the length of the slant side of the frustum, or $Ff = BG = AH$.



If A B be any diameter of an ellipse, and D E its conjugate; then if a semicircle be described on A B, and a perpendicular D G to A B, drawn from D or E, the area of the semicircle will be to the area of the semi-ellipse as C D to D G; and consequently two squares on these lines will have the same ratio to each other as the circle and the ellipse have. (Les. G. C. II., Pr. 29.)

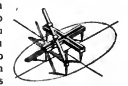
---

* The demonstration of the constructions in this Problem cannot be understood without a knowledge of the principles of linear perspective (See also Geometry, Appendix, *passim*); but as the constructions here employed are strictly within the limits of plane geometry, they are given because they offer a much easier and more accurate mode of obtaining the axes of an ellipse to which four given lines shall be tangents, than any deduced from the conic sections alone.



Let D C be the diameter of any circle, and let another, the diameter of which is just half D C, be supposed to roll within the larger round the circumference of it; the centre of the second will obviously describe a circle; any point in its circumference will describe a right line, a diameter to the larger circle; and any other points P, within or without the circumference of it, will describe ellipses, the axes of which will be the lines described by the extremities of the diameter in which P lies. (Note I 1.)

If six points be taken anywhere at pleasure in the periphery of an ellipse, or opposite hyperbolas, or parabola, and if the opposite lines joining them, by which a hexagonal figure is inscribed in the curve, be produced till they meet in three points, those points will be in a right line. (Les. G. C. II. P. 15.)



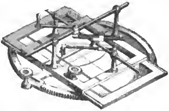
The *trammel*, or *elliptic compasses* as it is called, consists of a beam, like that of a beam compass, which carries a pencil at one end, and two sockets with cylindrical pegs which can be clamped at any distance from each other on the bar; these pegs move in two grooves at right angles to each other cut in the arms of a cross of brass, and by this motion the pencil describes an ellipse when the bar is turned round.

To adjust this instrument, the peg nearest the pencil is clamped on the bar at a distance from it, equal to the semi-conjugate axis of the ellipse intended to be drawn: and the farther peg is clamped at a distance from the pencil equal to the semi-transverse, that is, so that the distance between the pegs shall be equal to the difference of the two semi-axes: this distance being kept unaltered, the pegs are forced by the turning of the beam to move backwards and forwards in the vertical directions of the cross grooves, which correspond with that of the axes of the required ellipse, and the pencil necessarily draws the curve. (Note J J.)

The cross stands on four pointed feet, by means of which it is set in the direction of the axes, which are previously drawn at right angles to each other on the paper, the centre of the cross, or the point where the grooves cut each other, being thus exactly over the centre of the ellipse.

But since the distance between the pegs must always be equal to the difference of the two semi-axes, if this difference in the ellipse to be drawn, be greater than the length of the groove from the centre, the instrument cannot be used, and therefore no ellipse with an excentricity beyond this limit can be described with the trammel, and the arms of the cross prevent a small ellipse being drawn at all by means of it.

To the carpenter or cabinet-maker, who seldom require a very excentric or a small ellipse, the trammel is of considerable use, but these defects, together with the awkward mode of its application on the paper rendering it totally useless to the artist and draughtsman, an instrument acting on the same principle and intended to remedy its deficiencies, was invented by Mr. J. Clements; a complete description of this, with illustrative plates, will be found in the 36th vol. of the Transactions of the Society of Arts. We can here only give a general account of it.

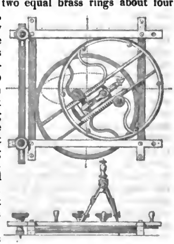


A square frame of brass carries in the middle of each side a pillar with a spherical head, through triangular holes in which slide two triangular bars at right angles to each other, and one raised above the other, its pillars being taller for that purpose. Axes passing through the centre of each bar are connected by a carriage lying between them; by a micrometer head and screw on which, the axes can be either brought vertically over each other, or removed to any distance within the limits of the length of the screw with the greatest accuracy. A second carriage and screw connected with the axes lies under the lower bar and carries the pencil or pen; this is also contrived so as to admit of being brought under the axes in the same vertical line: in this position the carriages with the pen can be turned round and the bars will not move in the sockets, the pencil only describing a *point* in the common centre of the two axes: if the pencil be now moved by the lower screw, which does not alter the axes, it will, if the carriage be turned round, describe a *circle* of any diameter from the point to the length of the screw, or about two inches radius: if the upper screw be then used to remove the axes to any given distance from each other, this being kept unchanged during the motion, the bars will advance and recede alternately in the sockets of the pillars, and the pencil will describe an *ellipse*, the bars corresponding to the cylindrical pegs moving in the grooves of the trammel. The conjugate diameter of the ellipse will be equal to that of the circle described before the axes were removed from their concentric position; the difference of the two elliptic diameters being given by the upper screw, which thus corresponds with the beam of the trammel.

For adjustment on the drawing board, the frame carrying the pillars is attached to and slides on another, having an inner cogged edge, worked by a pinion, as seen in the figure; this larger frame is carried by a circle which rests on the paper, with a cogged edge and pinion, thus enabling the centre of the upper frame to be adjusted accurately after it is first set down; by this means the centre of the intended ellipse is brought into the proper position on the drawing, while the right direction of the axes is given by the circle.

The minutiæ of the construction and adjustments of this instrument could not be explained without more detailed figures; but by means of these adjustments, and by a further addition which it is needless to describe, any ellipse, from a straight line to a circle of four inches, can be drawn to any given dimensions to the $\frac{1}{1000}$th of an inch, and this ellipse can be placed with equal accuracy in any part of the drawing, and may subsequently be divided either *perspectively* or *orthographically* into any number of equal parts. The plates given as examples of what it is capable of achieving, in the volume referred to, and which were drawn on the copper by means of the instrument, are a sufficient proof of the perfection of its construction; but this machine is too expensive to be in the hands of many, while the elliptograph of Mr. John Farey, though by no means so comprehensive or so complete, is sufficiently so for the purposes of most artists, and is simple and therefore cheap enough to be attainable by the majority.

This elliptograph consists of two equal brass rings about four and a half inches in diameter, lying one over the other; they are surrounded by a square brass frame, two opposite sides of which are at a small distance above, though fixed to the other lower two. Each ring has a projecting rim, that of the lower one overlapping the parallel sides of the frame between which it lies, and that of the upper ring overlapping the other sides of the frame. Each ring has also two parallel bars corresponding to each other, which slide between short pillars with expanded heads fixed in each ring respectively. By this construction the rings are kept close together, and yet each can slide between the parallel sides of the square frame in directions at right angles to each other when they are not concentric.



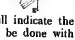
A pinion fixed to the lower ring works in a rack attached to one brass chord bar of the upper; by means of this the two rings may be fixed at any degree of excentricity of their centres within certain limits, the rack moving the upper ring between its parallel bars, while the other remains in the same place; the bars between which it lies preventing its motion in the *same* direction with the upper one.

Another pinion fixed in the upper ring works in a rack attached to a small frame, which thus slides between the bars of the ring, and carries a socket for one leg of a pair of *bow* compasses, the other having either a steel pen or a pencil.

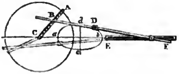
If the two rings are just over each other or perfectly *concentric*, and the frame carrying the pen so placed that this is over the centre of the two circles, then the rings being turned round between the frame by means of six small handles fixed into the upper circumference of the ring for that purpose, the pen or pencil will obviously describe a *point*, corresponding to the common centre of the rings; if by means of the last-mentioned pinion, the pen or pencil is moved between the chord-bars, the two rings being still concentric, the pencil will describe *a circle* of any diameter less than that of the rings, but if these are moved excentric by means of the other pinion, the compound motion of the rings advancing and receding in directions perpendicular to each other between the upper and lower parallel sides of the frame will cause the pen to describe an *ellipse*, the conjugate diameter of which will always be twice the distance of the point of the pen or pencil from the centre of the upper ring; and the difference of the two semi-axes will be equal to the distance of the two centres of the rings or of their excentricity. (See the Note J J.)

The whole frame is attached to another bar, on the under side of which are two small sharp pins which keep it steady on the paper, and it is held down by two fingers placed on two milled heads with screws which pass through rectangular holes in the extremities of the upper bars of the frame, which are extended a little beyond the others for this purpose, and lie over the detached bar just mentioned: in this there are two similar holes, and the screw acts on flanches which press the bars together and fix the frame to the detached bar; but when the screws are slackened, these holes allow the frame to be moved a small quantity in directions at right angles, so as to adjust it correctly to the situation where the ellipse is required to be on the drawing, the detached bar not being moved, after it is put down on the board; when the correct situation is found by trial, the milled heads are tightened and the instrument is fixed and held by the fingers of the left hand. To adjust the instrument, place it on the paper so that the detached bar, which is on the same level with the lower ones of the frame, shall be parallel to the shorter axis of the intended ellipse, and so that the centre of the two rings when concentric shall be over that of the ellipse: the pen being moved by the rack and pinion to one end of this shorter diameter, the rings are turned round; and if the point of the pen does not coincide with the other vertex of the conjugate, the frame is moved parallel to the detached bar, a quantity equal to half the error, and the pen is moved by the pinion for the other half; the trial is then repeated till the adjustment in this direction is made. The rings are then turned one quarter round, the rings are then moved excentric by the rack and pinion for that purpose till the pen is brought to the end of the transverse diameter, and an analogous adjustment to the former of shifting the whole frame in the direction perpendicular to the bar half the error and changing the excentricity of the rings the other half. Of course care must be taken in making this last adjustment not to alter the previous one made for the shorter diameter, but the power of doing this is soon attained by practice; and then the correctness and neatness of the curve, the simplicity and, consequently, cheapness of the instrument render it the most generally useful of any.

A simple mechanical application of the principle of these instruments, which will frequently be found of use to the draughtsman, must be noticed here.



A B, D E being the axes of an ellipse which is required; take a piece of flat paper having a straight edge cut; and mark on it the distance A C and C D from any point P to R and Q, then this edge being applied anywhere so that the points R and Q may always be in the lines of the axes, the point P will indicate the curve, and a dot may be made accordingly; if this be done with care, any number of points may be found with considerable accuracy, either to draw the curve through by hand, or for the purpose of making other constructions when the curve is not required.



A cheap and simple instrument is easily made, by means of which a curve very nearly approaching an ellipse can be described, and is employed for that purpose by some engravers and artists. A C is an arm or radius moveable on the fixed centre C; E F is a bar of wood or metal having a groove in it, in which slides a cylindrical pin, fixed near one extremity of the rule B F, the other end B is attached at any part of the radius A C by any convenient contrivance.

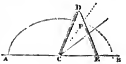
If now the radius A C turn round C, by which the end B of B F is made to describe a circle, the pin causing the other extremity to move backwards and forwards in the straight line of the groove; a pencil fixed any where in the rule B F as at D, will describe an oval curve which sufficiently approaches to a true ellipse, as not to be distinguished from it by an inexperienced eye.

The longer axis *a b* will always be equal to twice the distance from C to B, or the point where B F is attached to A C, wherever the point D may be between B and F: consequently the distance C B being made equal to half the given axis, the point P must then be moved on the bar till it pass through the vertex of the shorter axis *c d*; this adjustment is easily made by a trial or two.

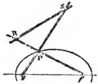
The longer the rule B F is, the more nearly will the curve resemble a true ellipse.*

Though the mechanical mode of describing an ellipse by means of a thread and pins, and explained in p. 78, would always be employed, when possible, as the simplest; yet the following methods of describing this curve by means of jointed rules may be occasionally useful to the mechanic.

1. Let C D E be a rule of equal legs jointed at D, and let the end C be fixed to the board by a pin on which it may turn as a centre, then a pencil fixed at any point P in D E, will describe an ellipse, if the end of the rule E be moved along a line A B for the transverse axis. This is a mechanical application of the second case, Prob. 56.—See Note D D.



2. Let the extremities of two rules be fixed to pins at F, *f*, which represent the foci, so that the rules may turn on them as centres; then another rule must be jointed to the others at R and S, placed so as to cross each other at P, F S, *f* R being made equal to the transverse axis, and R S equal to F *f*, then a pencil at P will describe the curve, when the rules are turned on F *f*. This is a mechanical application of the third construction in Case 1, Prob. 59.

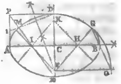


Carpenters and other mechanics frequently have recourse to the following methods of describing a figure nearly resembling an ellipse, consisting of segments of circles of different radii, passing through points which would lie in the true ellipse.

Let A B, D E bisecting each other at right angles, be the given axes of the proposed curve. Draw A P, D P at each extremity of these two axes, and parallel to them; bisect A P in 1, and

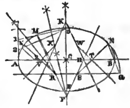
---

* If the rule were *infinitely* long, it would always move parallel to itself, and the point P would consequently describe a circle or a *true ellipse* of equal diameters.



draw E P, D 1 cutting each other in M; or if A C be bisected, E P must be drawn through the point of bisection; then the point M will be a point in the ellipse. (Prob. 56, Case 2.)

Bisect D M by a perpendicular cutting D E in F. Draw F Q perpendicular to, and equal to F D; from Q draw a line through B; and cut it in the point G, by an arc described from F, with the radius F D or F Q; draw F G cutting A B in H. Make A I equal to B H, and E K equal to D F, and draw lines from K and F through I and H. Then F, K, I, and H, are the four centres of the segments to be described with the radii F D, and H B; these will pass through M, and points corresponding to it in each quarter, and will form a figure resembling an ellipse, nearly enough for some practical purposes.



If A P and A C were each divided into three equal parts, and lines were drawn from E and D through the points of division; as shown in the figure, two points in the ellipse M, N, would be found by their intersections.

Find the centre F, in D E, of the segment to pass through D and M as before, draw F M cutting A B in I, and make C H equal to C I, C K equal to C F, and draw F H. Bisect M N by a perpendicular cutting F M in R, and through R draw R Q parallel to A B, cutting F H in S; make S Q equal to R M or R N, and draw Q B, from S cut Q B in the point N with the distance S Q, (R M or R N). Join S N, cutting A B in T, and make B V equal to A T. From K set off K W, K X on K H, K I equal to F S or F R, and then F, K, R, S, W, X, T, and V, will be the eight centres from which the segments are to be described with the proper radii, so that they shall pass through A, B, D and E, and through M, N and points in the other quarters corresponding to them: this figure will be more correct than the preceding.

In these constructions care must be taken to terminate each segment at the line joining its centre with that of the adjacent segment, so that there may be no angles in the curve. (See Prob. 48.)

When this factitious ellipse is made use of by the carpenter or mason, for the mould of an arch, the *normals* to the curve, representing the joints of the stones, must be drawn to the centre of each segment in which they occur.

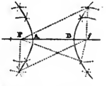
From all that has been said of the objections against these modes of drawing the ellipse, the artist will conclude that he must, in the greatest number of cases, trust to his eye and hand for delineating the curve, after having found a sufficient number of points: for this reason many problems have been given for this purpose, and for finding the true direction of the axis; when these are obtained, and the curve carefully traced by the hand of an experienced draughtsman, the defects in smoothness of line are of little importance in perspective drawings of instruments or buildings, the occasions on which the ellipse is mostly wanted.

§ PROBLEMS RELATING TO THE HYPERBOLA.

Problem 62.

To find points in the curves of the hyperbolas from different data.

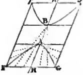
Case 1. Given the transverse axis A B, and the foci F *f*. (Geo. App. *passim.*)



Take any points in A B produced beyond F, then with the distance of each point respectively from the two vertices of the axis A and B, describe arcs from the foci as centres on each side of the axis, and the intersections will be points in the curve. (Geo. Ap. Prop. 21. Cor. 4. and Schol.)

Each assumed point in A B produced, will thus furnish two points in the curve of each of the opposite hyperbolas.

Case 2. Given any diameter A B, and a double ordinate to it F G, (Geo. Ap. Def. 17, p. 16.) cutting off the abscissa A H.

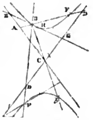


Through each extremity of F G, draw lines parallel to and equal to the abscissa A H; divide each half ordinate, and each of the parallels just drawn, into the same *number* of equal parts; then lines from the opposite vertex (B) of the diameter, to that at which the construction is made, drawn to each point of division in the ordinates, will be cut by lines drawn from the first vertex (A) to the corresponding points of division in the parallels to the diameter, in points of the curve. (Note K K.)

The same construction may be made for the opposite hyperbola with the corresponding ordinate and abscissa *f g*, B *h*, and if the conjugate diameter to A B be found (Prob. 63.) or be given the *conjugate hyperbolas* (Geo. App. Schol. to Prop. 19.) may be drawn by means of a similar construction. It need hardly be remarked that the same construction applies to the axis. In this case the ordinate will be at right angles, and the parallelogram will be a rectangle. (See Prob. 56. of the Ellipse.)

Case 3. Given the asymptotes, and any one point B, in the curve. (Geo. App. Props. 6 and 14.)

1. Through B draw lines, in any directions, at pleasure; but so as to cut both asymptotes, as A D, B F, E G, &c. &c.



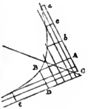
Make D F equal to A B, and G H equal to B E, and the same with the others; then the fourth point thus found in each line will be a point in one of the curves. (Les. G. C. I. Prop. 16.)

The equal segment in each line must be set off from the intersection of the one asymptote, in the *contrary* direction to that in which the point (B) lies from the other intersection.

If any of the points thus found be made use of as a new original point, and treated in the same manner, any number of new points may be found, or the former verified.

Since the point C in which the given asymptotes intersect is the centre of the curves, (Geo. App. Def. 15. and Prop. 14.) a line drawn through B and C, and produced till it is bisected in C, will be a diameter, the conjugate diameter to B C may be found by Prob. 63, and then by the construction points in the conjugate hyperbolas may be obtained.

The corresponding points in the opposite hyperbolas to any of those found as above, may be obtained by drawing a diameter through C, and the point in question; and making the other half of this diameter equal to, from C, the distance of the original point.



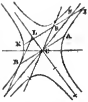
2. Through the given point B draw B A, B D parallel to the two asymptotes, and set off equal distances A *b*, *b c*, &c., equal to A C; and D *e*, *e f*, &c. equal to C D along each asymptote respectively; through these points of equal division draw lines parallel to the asymptote.

Make the parallel through *b* equal to half A B, that through C equal to one fourth of A B, and so on; in the same way make the parallel through *e* one half of B D, and the next to it one fourth of B D, the next one eighth of B D and so on, then the extremities of these parallels will be points in the curve. (Les. G. C. I., Prop. 18. Cor. 3.)

Problem 63.

Given a diameter A B, *to find the conjugate diameter and the asymptotes, the curves being given.*

Bisect A B in C, which will be the centre; draw any line K I terminated by the opposite hyperbolas, parallel to A B; and bisect K I in R; then C R being drawn is the indefinite conjugate

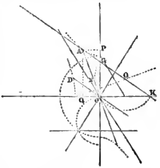


sought; divide K I in N, so that the rectangle K N, N I shall be equal to C B², (Prob. 7.) and make K L equal to N I; then lines drawn through L and N, and through C, will be the asymptotes.*

Through B draw a line parallel to C R; this will be a tangent to the curve at B, and the segment of this line intercepted from B by either asymptote will be equal to the semi-conjugate diameter to A B, and may be therefore set off from C on C R, each way for its vertices. (Les. G. C. 1., Prop. 16. Cor. 1. and 2. and G. App. Schol. to Prop. 19.)

Problem 64.

Given two conjugate diameters C A, C D *to find the axes, directrix, and foci, of the hyperbolas to which they belong.*



Through A draw a line parallel to C D; this line will be a tangent to the curve at A. (G. App. Prop. 19. Schol.) Make A N a third proportional to A C, C D. Bisect C N by a perpendicular, cutting the tangent at A, in O; from O as a centre with C O for a radius, describe a circle cutting O A in G, K; then C G, C K being drawn, they will be the axes; from A draw A P, A Q perpendicular to C G, C K, then a mean proportional between C G, C P will give the semi-transverse, and another between C K, C Q will be the semi-conjugate axis. (Les. G. C. I., Prop. 30.)

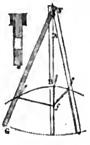
The hypotenuse of the right angled triangle formed by the two semi-axes set off from C each way on the transverse produced will give the foci. And if a circle be described on the transverse axis, and another on the distance between the centre C and foci, or on the excentricity, for diameters, the lines joining the two intersections of these circles will be perpendicular to the transverse, and will be the directrices. (Les. G. C., Prop. 6. and Schol.)

---

* K N, N I is equal to L I, N I, since K L is equal to N I by construction, and L I, N I is equal to C B². (Les. G. C. I. P. 18. Cor. 1.)

The foregoing are all the problems relating to the hyperbola that it is necessary to give in this work, it being a curve not often required either by the draughtsman or mechanic. Since most of the constructions relating to the ellipse are equally applicable to the hyperbola, attention being paid to the difference of the curves, any intelligent reader will be able to apply the rules accordingly, if he should have occasion for the analogous construction applied to the latter curve.

The following is a mechanical mode of drawing the hyperbola, and corresponds to that of drawing the ellipse by means of a thread, being founded on the same principle. (Geo. App. Prop. 21. and Schol.)



If A B be the transverse axis, and F *f* the foci, let a rule, so constructed that it can turn round a pin against the edge, be fixed at F, with one end of a thread made fast to a pin in the other focus, the other end being tied to the extremity of the rule at G, the whole length of the string being equal to the length of the end G from B and the distance B *f* in addition, then a pencil kept against the edge of the rule, so as to keep the thread stretched from *f*, will describe half an hyperbola if the rule be turned round F as a centre. To describe the other half, the rule must be reversed, as shown by the dotted line, and by changing the end of the rule to *f* instead of F, the opposite hyperbola may be drawn.

A piece of brass fixed across one end of a common rule, just projecting beyond the edges, with two small holes made through it close to those edges, and opposite each other, as shown in the figure, will be found useful for this purpose, a little stud may be fixed at the other end of the rule to fasten the string to.

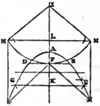
The description of an instrument contrived by Mr. R. Child for drawing hyperbolas, will be found in the "Mechanics Magazine" for January 2, 1830.

§ PROBLEMS RELATING TO THE PARABOLA.

Problem 65.

To find points in the curve of a Parabola from different data.

Case 1. Given the axis A K, the focus F, and the parameter D E, or the directrix L M.

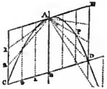


1. Bisect F L in A for the vertex, if the directrix is given; or make F A, A L each equal to one fourth of D E, that is F L equal to D F or F E, if the parameter be given, then L M drawn at right angles to the axis will be the directrix.

From any points I, taken at pleasure in the axis beyond L, describe arcs with the radius I F, to cut the directrix in two points M M, which will obviously be at equal distances on each side of L; from M and from F with the same radius describe arcs intersecting in a point on each side of the axis, which two points will be in the curve. (Geo. App. Pr. 21. Cor. 4) (Les. G. C. I., Prop. 1.)

2. Draw any ordinates at pleasure parallel to the directrix and parameter, and therefore perpendicular to the axis; then with the distance of the intersection of each ordinate, and the axis from the point L (as L K for example,) for a radius, from F as a centre, describe arcs cutting that ordinate in points in the curve as G, G, that is F G is to be made equal to L K.

If the parabola is large, it will be better to combine both of these methods, employing the first for points more remote, and the second for those near the vertex; or for verifying any point obtained by the other method, when the arcs cut each other too obliquely.

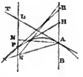


Case 2. Given any diameter A B, and a double ordinate C D. Through the extremities of the ordinate C, D, draw lines parallel and equal to A B; or draw a line through A parallel to C D, to cut the first parallels in segments equal to A B.

Divide each semi-ordinate into any number of equal parts; and each of the parallels to A B into the same number of equal parts, in 1, 2, &c., draw lines from A through the points of division in the parallels to A B as A 1, A 2; and through the points of division in the ordinate, draw lines parallel to A B; which will cut the former lines respectively in points in the curve. (Note L L.)

If the equal divisions are continued along the lines, produced points in the curve beyond the given ordinate will be obtained, as is shown in the figure.

This construction is obviously applicable to the axis, and its perpendicular ordinates.

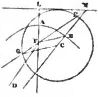


Case 3. Given a line A B for a diameter, and the tangent A T at its vertex, with any other point P in the curve.

Through P draw another diameter parallel to A B, and cutting A T in L. Set off any distance L N and A H equal to each other, but on contrary sides of the tangent; then lines drawn from A through N, and from H through P, will intersect each other in a point in the curve. (See Note L L.)

Problem 66.

The focus F *and directrix* L M *being given to find the intersections of the parabola with any given right line* C M.



Let the given line C M cut the directrix in any point M. Draw a line through M F; in the given line take any point at pleasure C, from which as a centre with a radius equal to its perpendicular distance from the directrix describe a circle, cutting F M in two points P and Q; draw C P, C Q* and draw F D, F E from F parallel to C P, C Q to cut the given line in D, E which will be points in the curve. (Les. G. C. I. Prop. 9.)

If the given line were parallel to the axis and therefore perpendicular to the directrix, the distance C M being the radius of the circle and F M drawn, the point P would coincide with M, and consequently a parallel to C P through F would also be parallel to C M the given line and would be the axis, and there could be no second intersection.

If the given line were inclined to the axis so as to cut it between F and L, one point of intersection would lie on each side of the axis; but if the given line produced cut the axis on the further side of L, then of course both points of intersection with the curve will lie on the same side.

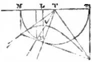
---

* C Q is not drawn in the figure to avoid confusion, and the point of meeting D of F D, M D is too far off to fall within it.

Problem 67.

The focus F *and directrix* L M *being given, to draw from a given point* T, *lines which would be tangents to the parabola.*

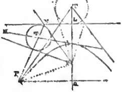
Case 1. If the given point T he in the directrix.



Draw T F; then lines bisecting the angles F T M, F T *m* will be the tangents; and if T M, T *m* be made equal to F T and lines be drawn through M and *m* parallel to the axis, they will cut the tangents in the tangential points.

If T were at L, the intersection of the axis and directrlx, these lines drawn bisecting the right angles and consequently at right angles themselves, will be the tangents, and an ordinate through F, perpendicular to the axis, will cut them in the tangential points.

Case 2. If the given point T be in the axis.



Make A Q in the axis equal to A T, and draw the ordinate P Q perpendicular to it, then with the distance F T, from F as a centre, cut this ordinate by an arc in P, R,* which will be the tangential points, and T P, T R will be the tangents required. (Les. G. C. I. P. 11. Cor. 3.)

Case 3. When the point T lies anywhere without the curve.

With the perpendicular distance from the directrix describe a circle from T as a centre, so that the directrix will be a tangent to the circle. From F the focus draw two other tangents to this circle cutting the directrix in M* and N, and M T, N T drawn from these points will be the tangents required.

The same construction may be employed in Case 2, T being in the axis, as is shown by the dotted lines, but the other is more accurate.

If the circle described on T, touching the directrix, should also pass through the focus, the point then lies in the curve; and if the length of the line F T be set along the axis towards the directrix, a line drawn through T and the point thus marked in the axis will be the tangent.

---

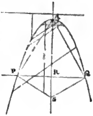
* The points R and M do not fall within the figure.

Problem 68.

A parabola being given, to find the axis, focus, and directrix.

Draw any two parallel chords as far apart as possible, so that each may cut the curve in two points; bisect them both, and a line drawn through the bisecting points will be a diameter.

Draw any other chord P Q at right angles to this diameter, and bisect it in R; then a line through R, parallel to the diameter already found, will be the axis, and the point A where it cuts the curve is the vertex.



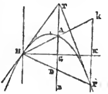
Join A Q, and draw Q S perpendicular to it cutting the axis in S; one fourth of R S, set off on the axis each way from A, will give the focus and the intersection of the directrix with the axis. (Les. G. C. I. Prop. 29.)

The focal ordinate to the axis is the *parameter* to the axis, and the ordinate to any diameter drawn through the focus is the *parameter* to that diameter.

Problem 69.

To draw an ordinate to a given diameter A B, *through a given point in the curve* H.

Through H draw H K perpendicular to A B, cutting it in G, and make G K equal to H G; or through H and A draw a line and make A *k* in it equal to H A. Then a line through K or *k* parallel to A B will cut the curve in a point P, and H P being drawn it will be the ordinate required. (Geo. II. § 4. Pr. 29.)



If the given diameter be the axis, a line through H perpendicular to it will be the ordinate itself.

Problem 70.

The parabola being given, to draw a tangent to it from any given point or parallel to a given line.

Case 1. If the given point P be in the curve.

1. Through P draw an ordinate P H to any diameter A B

(preceding Prob. and figure), and set the length of the abscissa D A along that diameter produced from A to T; then a line drawn through T and P will be the tangent required. (Leslie, Observ. to Def. 19. and Prop. 20, Cor. 3.)

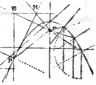
If T H be drawn from T through H the other extremity of the ordinate, it will be a tangent at H.

2. Through P draw a diameter, to which draw an ordinate through any point at pleasure; then a line through P parallel to the ordinate will be the tangent required.

As the farther the ordinate is from the vertex P, the longer it will be; so much the more correctly can the tangent be drawn parallel to it.

3. Draw P F to the focus and draw F M perpendicular to P F to cut the directrix in M; then P M being drawn will be the tangent. (Les. G. C. I. Prop. 9. Cor. 3.)

4. Through P draw a diameter, or a line parallel to the axis, to cut the directrix in E. Join E F, then P M drawn perpendicular to or bisecting E F, or bisecting the angle E PF, will be the tangent at P. (Les. G. C. I., Prop. 11.)

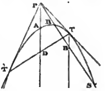


If the angle formed by a line drawn to the focus, and the diameter from any point in the curve be bisected by a line, this will be a normal to the curve at that point, and will be perpendicular to the tangent.

In this way the joints of the stones of a parabolic arch are drawn by the carpenter and mason.

Case 2. If the point P be not in the curve.

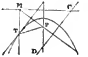
1. Through P draw any line to cut the curve in two points R, S, and make P B a mean proportional between P R, P S, then the vertex T of a diameter drawn through B will be one of the tangential points. (Note M M.)



2. Through P draw a diameter P D and make the abscissa A D equal to P A. Through D draw an ordinate to A D which will cut the curve in the two tangential points T, T'. (Case 1. 1.)

Case 3. If the required tangent is to be parallel to a given line C D.

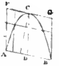
Through the focus F draw a line perpendicular to C D, meeting the directrix in M; then a diameter, or line parallel to the axis, being drawn through M cutting the curve in T, T will be the tangential point; the tangent required must be drawn through it parallel to



the given line, C D, and it will bisect F M and the angle F T M. (Les. G. C. I. Prop. 10.)

Problem 71.

To find a rectilinear figure equal to the space included by a parabola and any straight line cutting the curve in two points.



Draw C D the diameter to the given secant A B as an ordinate, and complete the parallelogram A B F G by parallels to the diameter C D, from A, B, and by the tangent at C parallel to A B.

Divide A F or A B into three equal parts, and draw a line through one point of division parallel to the other sides, thus forming a parallelogram equal to two thirds of A B F G which will be equal to the area of the parabola.

A rectangle equal to the area is made by taking two thirds of the perpendicular to A B from C for one side, and A B for the other.

A parabola may be readily observed in a stream of water flowing out of the tap of a barrel, or in the jet d'eau of a fountain.

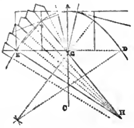
To draw a parabola mechanically, one side of a drawing-board being assumed as the directrix, or as a line parallel to it, a pin must be fixed in the board as the focus, and a T square applied to that side of it. A thread being then fixed to the pin and to the extremity of the blade of the square, will cause a pencil held against the edge of the square to describe a parabola, provided the thread be kept stretched, as the square is moved along the side of the board.

When the edge of the blade touches the pin, the part of the thread from the pin to the pencil, which will then be in the vertex of the axis as represented by the edge of the blade, will be double, and therefore will be equal to half the distance of the focus from the directrix.

The square must be reversed to draw the two halves of the curve, as the pin will not allow of its passing to draw them continuously. A line parallel to the edge of the board against which the square was moved, and at precisely the same distance from the vertex that this is from the focus or pin, will be the true directrix to the curve as thus drawn.

A curve resembling a parabola may be drawn composed of segments of circles in a manner analogous to that in which an imitation of the ellipse was produced.

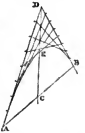
Let A C be the axis of the proposed parabola, and D E an



ordinate to it. Find one or more points in the curve M (by Prob. 65.) Join A M and bisect it by a perpendicular cutting the axis in G for the centre of the segment at the vertex. Draw M G: M and the next point E being joined and bisected by a perpendicular, this will cut M G produced in the centre H, for the next segment, and so of as many as may be thought requisite. If this construction be used by masons for an arch, the joints of the stones must be drawn to the centres of each segment, as was directed for the ellipse.

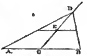
The property of the parabola, that a third tangent cuts off proportional parts from two others (Les. G. C. I. Pr. 20. Cor. 4.) presents an easy, and generally applicable, method of describing the curve.

Let A B be any line bisected in C by a line C D forming with it any angle; in C D take any point D and bisect D C in E. Join D A, D B and divide both into any, the same, number of equal parts. Join the points of division in succession from A to D with those from D to B, and these lines will all be tangents to the parabola, of which C D is a diameter, A B a double ordinate, and E the vertex to the diameter.



If the number of divisions, and consequently of the tangents, be sufficiently numerous, the curve will be formed by the portions of each tangent intercepted by those adjacent to it, and the small angularity may be removed by drawing the parabola by hand subsequently.

By these means a parabola can be made to pass through any three given points A, B, E, by joining the two extreme points A and B, and bisecting A B in C; draw C D through E and make D E equal to C E, and join A D, D B, and proceed with the construction as given above.



A line from any point of division in A D or D B parallel to C D will obviously be a diameter, and its vertex will be accurately defined by the intersection of the tangent which forms the portion of the curve at the part through which it passes; and as a line parallel to any of the tangents will be an ordinate and may be equally well defined at its extremities, many constructions may be made as accurately as if the curve were perfectly continuous.

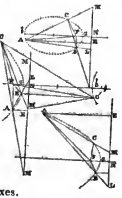
Thus the normal to the curve for the joint of the stone of an arch may be drawn vertical to any of these tangents at the tangential point of it.

Architectural draughtsmen make great use of this mode of drawing a parabola in the construction of Gothic arches, which are frequently made to consist of two equal half parabolas, their point of springing being the vertex of a diameter, and the straight side of the window or door forming a tangent to the curve at that point.

Problem 72.

Given three points A, B, *and* C *to find the axes and directrices of the conic section that shall pass through them, having a fourth given point* F *for its focus.*

Join F A, F B, and F C, and draw a line through any two of the points as A B; divide A B externally in L, so that F A : F B :: A L : B L (Plane Geo. Pr. 12.) through the remaining point C and either of the others A or B, draw a line and divide it externally in M, so that F C : B F (F A) :: C M : B M (A M); then a line through L and M will be the directrix, and a line through F perpendicular to L M will be the principal axis.



From either of the points as A draw A E perpendicular to L M, or parallel to the axis F N, and divide F N internally and externally in *a* and *b* in the ratio of F A : A E, that is so that F*a*:*a*N :: FA:AE and F*b*:*b*N :: FA:AE; then *a* and *b* will be the vertices of the axes.

1. If F A be less than A E, F *a* will be less than *a* N, and F *b* less than *b* N, and the curve will be an ellipse.

2. If F A be greater than A E, F *a* will be greater than *a* N and F *b* greater than *b* N, and the curve will be an hyperbola. The division of A C or B C must be internal if the point C′ is in the opposite hyperbola.

3. If F A be equal to A E, F *a* will be equal to *a* N; but there can be no external point of division, or *b* will lie at an infinite distance in F N and the curve is a parabola.

If F A, F B, and F C were all equal, the curve would be a circle and the directrix L M will be infinitely distant. (Les. G. Cur. II. Pr. 18.)

§ THE CYCLOID AND EPICYCLOID.

If a circle revolve on its centre so that a point in its circumference makes one revolution, in the same time that the centre of the circle describe a straight line equal to that circumference, the point will describe a curve called a *cycloid*, or sometimes a *trochoid.*

Thus if any point in the circumference of a carriage wheel were marked, it would describe the curve mentioned while the wheel rolled along a flat road, for the centre of the wheel would clearly move in a right line equal to the periphery, while the point made one complete revolution.

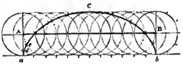
The circle, by the combined motion of which the curve is produced, is the datum for describing the curve geometrically, and is called the *generating circle.*

The right line on which the generating circle is supposed to roll, and which is equal to the circumference of it, is called the *base*, and a perpendicular to it from the central point, which will divide the cycloid into two equal parts, is the *axis* of the curve.

A line parallel to the base terminated by the curve and which is obviously bisected by the axis, is a *double ordinate*, and the corresponding segment of the axis the *abscissa.*

If the generating circle be described on the axis as a diameter, and an ordinate to the curve be drawn, the arc of the circle cut off from the vertex by this ordinate is called the *corresponding arc* to that of the curve.

1. The periphery or length of the circumference must be found by multiplying the diameter of it by 3·1416, &c., or by taking this number from any scale of equal parts, to which that diameter is the unit.



2. Draw a right line A B equal to the circumference of the generating circle thus obtained, and divide it into any convenient number of equal parts according to its length, as for example twelve.

3. On each point of division describe a circle equal to the generating circle, and draw $a\,b$ a tangent to all these and therefore obviously parallel and equal to A B, and divide $a\,b$ into equal divisions in order to obtain the true tangential point of each circle.

4. Divide the semicircle into half the number of equal parts

that A B was divided into, or, which is the same, divide the whole circumference into the same number of equal parts that A B was divided into, and from each tangential point on *a b*, from *a* and *b* in succession, towards the middle point, set off the chords of one, two, three, &c., parts of the circumference respectively, on the circumference of the corresponding circle described on the points of division of A B, and these points will lie in the curve of the cycloid, which may be drawn by hand through them. For it is obvious, from the description of the mode in which the cycloid is generated, that supposing the fixed point in the circumference of the generating circle to be at *a*, while its centre moves from A to the first point of division in A B, that the point will, by the revolution of the circle on its centre, come into the situation *d*, on the circle described on the first point of equal division from A to B, and when the centre of the circle comes to the second point of division in A B, the point in its circumference will come to *e*; and so on till the circle has performed half its course or revolution, when the point will be at *c*, since the half of A B is equal to the half of the circumference of the circle.

And in thus following the point, it will be found in points in the corresponding half of the curve in succession till it comes to the tangent point in *a b* again at *b*, while the centre has moved from A to B, or in a line equal to the circumference of the generating circle.

And if the circle were supposed to move on in the same line, a similar and equal cycloid would be again described by the point.

If the cycloid is large, it may be necessary to find points intermediate to those near the vertex, by bisecting the divisions on A B and describing the circle on these new centres, the division of the generating circle being also bisected and the chord being set off on the circumferences.

The points *d*, *e*, &c., between *a* and *c*, and the corresponding points between *c* and *b*, could also be found by drawing parallels to A B through the points of equal division on the circle, as is obvious from the figure, but this mode would not be so productive of accuracy as that given.

If the fixed point, instead of being in the circumference of the generating circle, were taken within it, a curve called the *prolate* or *inflected cycloid* will be produced, the successive portions of which formed by the continued motion of the circle will together form an undulating curve; and if the point were taken without the circumference, the *curtate* or *contracted cycloid* is produced, which has loops formed at each successive recommencement of the revolutions. If it were required to draw these modified cycloids, the line A B must be made, as before, equal to the periphery of the generating circle and divided into equal parts, but on these points of division circles must be described with a radius equal to the distance of the fixed point from the centre of the generating circle, and the chords of the equal divisions of these circumferences being set off as before, the points will be obtained.

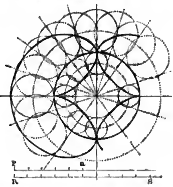
If the centre of the generating circle, instead of moving in a right line, were supposed to move in a circle; or if the generating circle were supposed to roll round the circumference of another circle instead of rolling on a right line, then the given point would describe the curve called the *epicycloid* or *epitrochoid*; if the generating circle roll round the convex circumference, the curve produced is called the exterior epicycloid; if it roll round the concave circumference, or within it, the curve is called an interior epicycloid.

1. The circle round which the generating circle revolves, and this latter being given, a circle must be described with a radius equal to the sum or difference of their radii according as the generating circle rolled round the outside or inside of the other: this will obviously be the circle described by the centre of the generating one during its course.

2. Draw a right line P Q equal in length to the periphery of the generating circle, and divide it into any number of equal parts. A second right line R S must be made equal to the circumference of the circle round which the generating circle rolls, and divisions equal to those of P Q must be set along it, and the number into which it is thus divided observed.

3. Divide, by trial, the circumference of the circle described by the centre of the generating circle into as many equal parts, and a fractional part as the right line R S contained of the divisions of P Q, and draw radii through the points of division, and on each of these describe the generating circle, the points where the radii cut these circles obviously correspond to the tangential points in the line *a b* in the figure of the cycloid, and from them the chords of the equal divisions of that generating circle must be set in succession, to obtain the points in the curve of the epicycloid, in a manner analogous to that by which those of the cycloid were obtained.

If the circumferences of the two given circles were commensurable, the right lines P Q, R S, or the equals of those circumferences, would be also commensurate, and the magnitude which divided P Q into any assumed number of equal parts would also divide R S into some definite number of equal parts without remainder. On this supposition the fixed point in the circumference of the generating circle will describe, in its progress round the other, one or more epicycloids which will recur over again in the same places, on the commencement of a second complete revolution; but if the circumferences or the equivalent lines PQ, RS are not commensurable, then the second series of curves will be like the first, but will not coincide with them.



If the generating circle be of the same diameter as that round which it rolls, then the point will describe one epicycloid terminating at the same point it commenced on. In this supposition, of course no interior epicycloid could be generated, or it will coincide with the circles themselves.

If the generating circle be smaller than the other, more than one epicycloid will be produced; and if the circumferences are commensurable, the curves will terminate in the same point as they began, or will return into themselves; and if the generating circle be larger than the one round which it revolves, the curve will not be completed while the generating circle revolves once round the other, and will form a loop at each recommencement of the epicycloid. In this case also of course there can be no interior epicycloid.

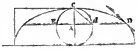
If the radius of the generating circle be just one half of that of the other, then the interior epicycloid will be a straight line or the diameter of the circle. (Les. G. C. III. Prop. Epicyc. Prop. 5. Cor.)

If the fixed or tracing point be taken within or without the circumference of the generating circle, modifications of the curve analogous to those produced by the same conditions in the cycloid will be produced, and which may be conceived without further exemplification.

The following properties of these curves may be of occasional use to the draughtsman.

In the cycloid.

1. An arc of the cycloid measured from the vertex is equal to twice the corresponding chord of the generating circle, that is $DC = 2\,dC$ (Les. G. C. III. Cyclo. Prop. 3.) and the whole curve is equal to four times the diameter of the generating circle.



2. If a tangent be drawn through the vertex parallel and equal to the base, and the rectangle be completed by perpendiculars to

these parallels from the extremities of the base, then the space comprised by the curve and the three sides of the rectangle is equal to the area of the generating circle, and the space comprised by the cycloid and the base or fourth side of the rectangle is equal to three times the area of the generating circle.

3. The tangent to a cycloid at the extremity of any ordinate is parallel to the chord of the corresponding arc drawn from the vertex; that is, the tangent at D is parallel to C d.

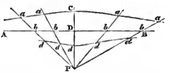
In the epicycloid, a tangent to the curve at a given point is perpendicular to the chord drawn from the point of contact of the two circles to that given point, when the generating circle has brought the fixed point in its circumference to the situation where the tangent is drawn.

These curves are important to the engineer from their having been proposed as the form of the teeth of cogged wheels and racks.

If the balls which represent the earth and moon in the machine called an Orrery be taken off and pencils put in their places, and a sheet of stiff flat pasteboard fixed on the place of the sun, one pencil will describe a circle and the other an epicycloid; but from the erroneous proportional distances of the planets in the machine, the epicycloid will be very unlike the kind of curve really described by the moon in space.

THE CONCHOID.

If a point P, and a straight line A B, be given; and any lines be drawn from P to cut A B; and equal segments a b be set off on them from the points of intersection b; the curve passing through the extremities a of these equal segments will be a *Conchoid*, called from its inventor, that of Nicomedes.



When the equal segments are set off on the other side of A B, to that on which the point or *pole* P lies, the curve is termed the *superior conchoid*.

And when the segments are set off on the same side of A B as P, the curve is called the *inferior conchoid*.

The line P C perpendicular to A B is the axis; and C D, or the segment, the diameter; the line A B is the directrix. To draw the curves, it is therefore only necessary to draw a sufficient number of lines through P, and make the segments *a b* equal, and the curve must be drawn by hand through the points *a*.

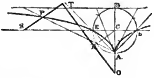
If the segment *a b* of the radiating line of an inferior conchoid be larger than the perpendicular distance, P D, of the pole from the given line A B, the curve will form a loop and pass through the pole P twice, or P will be the point where the curve crosses itself.

This curve is of importance to architects from its being that best adapted for the contour of the *shaft* of the column.

In this application of it, the axis of the shaft is made the directrix of the superior conchoid; the axis of the curve is taken just above the mouldings of the base, and of course at right angles to the directrix or axis of the column; and consequently the semi-diameter of the shaft at that part is the *diameter* of the curve, the distance of the *pole* P being made equal to the length of the shaft.

Thus the shaft of the column is a solid formed by the revolution of a conchoid about its directrix.

THE CISSOID.



Let A B be a diameter of the circle A B D E, and let a tangent be drawn through the extremity B: from A draw any number of lines at pleasure to cut the tangent, and from the points of intersections set off towards A, the length of the chord intercepted by the circle from A in each line respectively; then a curve drawn to pass through the points P thus found, will be that called the *cissoid* of Diocles, so called from its discoverer.

It is obvious from the construction that the curve will have two equal and similar branches arising from A, and passing through the extremities D, E, of the diameter parallel to the tangent, and also that the tangent is an assymptote to the curve.

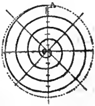
If the line A P be drawn through any point P in the curve, cutting the circle in R and C R joined, and a line through P parallel to C R be drawn, cutting the diameter D E produced, in S, then S P will be equal to the radius of the generating circle. And if P T be made also equal to S P or C R the radins, and T O be drawn perpendicular to S T, to cut A B produced in O, then O A will be equal to A C or to the radius.

This property affords a mechanical mode of drawing the curve: for if S T, the side of a triangular rule S T O, were equal to the diameter of the generating circle, and a pin were fixed at O, then a pencil at P in the middle of the side S T, would describe the curve; the edge of the rule O T being always moved against the pin, while the angle of the rule S moved along the perpendicular diameter S C.

The cissoid is a curve of great importance to mathematicians from its affording a solution to several curious problems: but it is never used in the arts, and therefore the mode of describing it is all that is necessary to be given in this work.

OF THE SPIRAL.

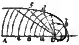
If from a given point, any number of lines forming equal angles be drawn, and the length of each increases in succession by an equal quantity, the curve that passes through these points is that called *Archimedes' Spiral.*



If the spiral is to pass through a given point A, and to consist of a given number of turns from the centre C, as for example four; draw A C, and divide it into four equal parts. Draw any convenient number of lines from C, forming equal angles with each other and with A C; subdivide each fourth part of A C into the same number of equal parts that there are angles formed by the radii round C. Then set off the length of the line A C, diminished each time in succession by one subdivision, on each radius from the centre C, and the spiral must be drawn by hand through the points thus found, and will commence from A, and make four revolutions terminating in the centre C.

It is obvious that, if the radius, instead of increasing in length by an equal quantity, as it is supposed to revolve round C, were to increase by any other law, as, for example, in a geometrical ratio, then different spirals would be produced; and if, combined with this variation, the angles formed by the radii were to vary by some other law, the different species of spirals might be still further increased.

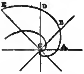
The principal spirals besides the above are:



1. That termed the Hyperbolic or Reciprocal, which is a spiral passing through the extremities of any number of arcs of circles of equal length, measured from a given right

line C A, and described from the centre C; that is, the arcs *a b*, *c d*, *e f*, &c., described at any distances at pleasure from C, are equal.

To make them so practically, any small quantity at pleasure must be set off along them by spring dividers, the same number of times.



2. The Logarithmic, in which the radii make equal angles, and the spiral cuts them all at an equal angle: the length of the radii increases in a geometrical progression, that is C A : C B : : C B : C D : : C D : C E, &c.

Spirals are found in a variety of objects, produced naturally in shells, as that of the snail, and an infinity of others; in a watch spring; in the sparks from the common fire-work called a Catherine wheel; &c. &c.

The spiral is of interest to the architect, from its being the form of the ornament of the capital of the Ionic column called the *Volute;* and a great many modes have been suggested by different artists, of producing spiral lines for this purpose, by portions of circles described from different centres, but all these curves thus produced are deficient in the grace of form which characterizes the volutes of the columns of the ancients, and which can only be copied by a spiral drawn very carefully by hand.

An instrument has been contrived by Mr. Jopling, by means of which a great variety of curves may be drawn, applicable in different arts.

It is a frame composed of jointed rules, capable of having the centres of motion set at any relative distances from each other; and by the various compound motions thus produced, the curves are described.

The general principle of Mr. Jopling's instrument may be understood from imagining various combinations of either the right line or circle substituted for the lines and equidistant points by which the elliptic motion of the common trammel is produced: thus if two pins were made to move in the circumferences of two circles, or in a right line and a circle, instead of in the cross grooves of the trammel, a pencil carried by a beam to which the pins were fixed would describe a compound curve instead of an ellipse; and if the distance between the pins were made to vary during the motion, a new form would be produced. The object of Mr. Jopling's instrument is to allow of these various combinations being practically made use of; and it is both simple and ingenious. He has in his professional pursuits applied the curves thus obtained on several occasions with great success.

In a Scholium to Prop. 4. of Sect. 10. Book III. of the Geom. of Curves, Professor Leslie described an instrument by which the *Catenary* may be drawn; and from the circumstance of this curve not admitting of being geometrically described, and from its importance in the construction of suspension bridges, this simple and elegant instrument is of great utility.

Descriptions of many instruments for drawing various curves may also be found in "Adams's Geometrical and Graphical Essays."

LINEAR PERSPECTIVE.

LINEAR PERSPECTIVE.

INTRODUCTION.

LINEAR PERSPECTIVE is an application of Geometry to the Art of Drawing, by which the outline or form of a particular class of objects can be delineated as they appear to the eye.

Objects are seen, as is well known, by means of rays of light proceeding in right lines from every point of their surface; these rays enter the eye, and produce on the *retina* an inverted image, or picture, of the original. If a plane be supposed placed between any such object and the spectator, it would intercept these rays; and if the points of intersection were coloured, so as to resemble the corresponding points of the object, an image or picture of this would be produced on the surface, such as it appeared to him in that position.

To explain this more clearly, let the spectator be supposed to look at any object, such as a building, through a window. If, always keeping his eye steadily in the same place, he marked on the glass the different *lines* by which the various parts of the original *seemed* separated from each other, or bounded, he would obtain the *outline* as it is termed of the building, as it appeared to him from that point of view.

And such an outline or image, even when removed from its original situation, would *partly* convey an idea of the building to any other person who placed his eye to view this image in the same relative situation with respect to it; because the rays from this outline would form, on the retina of his eye, an image similar to that which would be produced by the rays which would have proceeded from the original object, and will therefore affect his mind in the same way, or produce a similar idea.

But since it is impracticable to obtain such an outline by interposing a plane in the manner described, the artist has recourse to his hand and eye, to copy on a convenient surface, such as paper or canvass, as accurately as he is able, the apparent *contour* of the original object he wishes to draw, provided this do not consist of, or cannot be reduced to, geometrical forms; as is the case with the human figure, trees, animals, or most natural productions. To effect this, is one principal part of the art of drawing.

But the outlines of those objects—as buildings, machinery, and most works of human labour—which consist of geometrical forms, or which can be reduced to them, may be most accurately obtained by the rules of *linear perspective,* since the intersection, with an interposed plane, of the rays of light proceeding from every point of such objects, may be obtained by the principles of *geometry.*

An acquaintance, therefore, with this art is indispensable to the architect, engineer, and all those who require outlines of such objects in their different pursuits.

Having thus generally shown the connexion between the art of drawing and linear perspective, the principles of the latter will be now explained as a part of geometry, in pursuance of the plan of this work.

Let the place of the eye, therefore, be considered as a *geometrical point,* the rays of light as *right lines* producible indefinitely, and the surface on which the outline is formed, a *plane* indefinitely extended.

Of all such rays there must be a series which, if produced beyond the object, would be *tangential* to its surface; that is, each such ray would touch it in one point only, and would not be impeded in its prolongation by the solidity of the body. The intersection of this series by the interposed plane would obviously give the apparent exterior outline of the original.

Where two portions of the surface of an object, which are not continuous, meet, a line is apparently formed; and the rays from every point of such lines will, by their intersection with the plane, form lines in the *image* or representation, corresponding to the original lines; and the whole image thus produced is termed the *perspective projection* of the original.

Each such series of rays forms a pyramid,* the vertex of which is the point in which the eye is supposed to be placed; the form of the pyramid being determined by that of the portion of the original object which constitutes its base: and in the limitation of the class of objects to which the rules of perspective are applied, this form will always be a geometrical figure, or at least reducible to such.

In the first part of the following treatise, such original objects will alone be considered as are bounded by planes forming by their intersections right-lined figures; or geometrical figures, bounded by right lines, will themselves be considered as original objects.

---

* This is not always a pyramid, in the *geometrical* sense of the word; since, from the variety of forms of its base, some of its sides may be plane triangles, and some conical surfaces.

DEFINITIONS.

Def. 1. The *point* in which the rays proceeding from an original object or figure meet, is termed the *vertex.*

This, from its being the place of the spectator's eye, is also called the *point of sight*, and is the vertex of the imaginary pyramids before mentioned.

Def. 2. The *plane* on which the image of any object is formed by the section of these rays, is called the *plane of projection.*

From the application of perspective to painting, this is also called the *plane of the picture.*

Def. 3. A plane passing through the vertex, parallel to the plane of projection, is termed the *vertical plane.* (See Geom. App. §. 1.)

Def. 4. The plane in which any original point, line, or figure, considered as an object, lies, is called the *original plane* of that point, line, or figure; and the different planes which form the surface of a solid are called *original planes* of that solid, and are supposed to be indefinitely extensible.

Def. 5. The plane which passes through any original right line, and through the vertex, is termed the *projecting plane* of that line.

The perspective *projection* of an original right line is therefore the intersection of the projecting plane of that line with the plane of projection, and this intersection is always a right line. (Geom. App. 1, Pr. 2.)

Def. 6. The intersection of any original plane with the plane of projection is called the *intersecting line* of that plane; and its intersection with the vertical plane is called its *station line.*

Def. 7. The point where a perpendicular from the vertex meets the plane of projection, is called the *centre of the picture,* and the length of this perpendicular is the *distance of the vertex.*

It is also called, for reasons before given, the *distance of the picture.*

Def. 8. The right line passing through any original point and the vertex is called the *ray* of that point.

The perspective projection of an original point is the intersection of its ray with the plane of projection.

1. An *original plane* can only have three positions relatively to the *plane of projection.*

If it be *parallel* to it, it will also be parallel to the vertical plane, and it can consequently have neither *intersecting line* nor *station line.* (Def. 6.)

If it be perpendicular to the plane of projection, it will be also perpendicular to the vertical plane.

If it be *inclined* to the plane of projection, it will be inclined in the same angle to the vertical plane; and in this and the preceding case, its *intersecting* and *station lines* will be parallel to each other, these being the intersections of parallel planes by a third plane. (Geom. IV., Pr. 12.)

2. If an original plane pass through the *vertex*, the *rays* from any points in it will lie wholly in that original plane, and these rays will obviously cut the plane of projection in the intersecting line of that plane. (Def. 6.)

Consequently, the *perspective projections* of all original lines, lying in such an original plane, will coincide with its *intersecting line*, or the projecting planes of all such lines will coincide with the plane itself. (Def. 5.)

3. But if the original plane do not pass through the *vertex*, all lines in it will have separate *projecting planes*, and consequently separate *perspective projections*.

And it necessarily follows that no original line lying in such a plane can pass through the vertex.

Let an indefinite right line, lying in an original plane, which is supposed not to pass through the vertex, be next considered.

4. If this original line be *parallel* to the intersecting line of the plane in which it lies, it will also be parallel to the station line, and can consequently never intersect either the *plane of projection* or the *vertical plane*. (Geom. IV. § 1. P. 10.)

5. But if this line be not parallel to the intersecting or station line of the plane in which it lies, and consequently not parallel to the *plane of projection* and *vertical planes*, it will cut these.

Def. 9. The point in which an *original line* cuts the plane of projection, is the *intersecting point* of that line; and the point in which it cuts the vertical plane, is its *station point*.

6. And it is obvious that the *intersecting* and *station points* of any original line, will be in the *intersecting* and *station* lines respectively of the original plane in which it lies.

7. As the rays from all points in any original line lie wholly in the *projecting plane* of that line, (Def. 5, 8.) the *perspective projections* of all points in the original line will be points in the perspective projection of it. (Def. 5.)

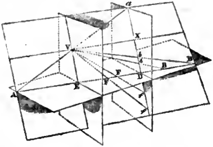
8. Let V[*] be the vertex, A B an original line, D being its in-

---

[*] In these diagrams, it is necessary to express a plane by *four* lines apparently limiting it; but the reader must constantly bear in mind, that *all* planes are supposed indefinitely extended in every direction.

The following notation will be invariably used, unless expressly stated otherwise. V is the vertex, C the centre of the picture, (Def. 7.) Capital letters will be used to express points in an original line: thus A B means an indefinite right line in which two points are marked by those letters; and the same letters, small, will be applied to the *perspective projections* of those original points, and therefore will also express the indefinite perspective projection of the original line A B: *a b* is therefore the projection of the line A B.

tersecting, and E its station points; *a b* will be the *perspective*



projection of A B, since it is the intersection of the *projecting plane* of A B with the *plane of projection*.

The point E, or the *station point*, is the only one in A B that can have no projection*. (Geom. App. Pr. 1.) The point D, or the *intersecting point*, is identical with its projection, it being common to the three planes.

Through V let a line be supposed to pass parallel to A B, and therefore lying in its *projecting plane;* and let it cut the *plane of projection* in X, which it will necessarily do in *a b*, the projection of A B.

Then the projections of all such points as B, lying in A B beyond D from E, will lie in the segment D X of the projection of A B; and the farther from D, B is situated, the farther from D or the nearer to X will its projection *b* lie: X, therefore, may be considered as the projection of a point in A B at an infinite distance from D.

Points such as F, lying between the *intersecting* and *station points* of A B, will have their *projections* in *a b* beyond D from X; and the nearer F lies to E, the farther from D will its projection *f* be: the projection of E is consequently at an infinite distance from D in *a b*, or it has no projection at all (Geom. App. Prop. 2, and Cors.) as was before stated.

The points, A, lying beyond E from D, have their projections in *a b* beyond X from D, and the farther A is situated from D beyond E, the *nearer* its projection approaches X: X may consequently be considered as the projection of a point in A B, infinitely distant from D, in the contrary direction to the infinitely distant point (B) of which it was before stated to be the conventional projection.

---

* In order to avoid unnecessary repetition, the word *projection* used alone will be understood to mean perspective projection: the term *image* will be also used to express the same thing.

Google

Def. 10. The point (X) where a line passing through the *vertex* parallel to any original line, cuts the *plane of projection*, is called the *vanishing point* of that original line; and this point may be considered as the projection of two points in the original, infinitely distant from the intersecting or station points, in opposite directions.

Def. 11. The line (V X) through the vertex parallel to any original line by which the *vanishing point* of this last is determined, is called the *radial* of that original line.

9. As the *radial* of any original line will also be parallel to all other original lines to which the first is parallel, it will be the common radial of them all; and the vanishing point determined by that radial will be the common vanishing point of all lines parallel to the first.

Hence, also, since the *radials* of original lines are parallel to them, the radials form the same angles at the vertex that the originals make with each other.

Suppose any number of original lines parallel to each other, but lying in different planes. The rays from all these lines will lie, as has been already stated, in their several *projecting planes*. Now since all these projecting planes pass through the original lines, and through the vertex, their common intersection will be parallel to those original lines, and will pass through the vertex. (Geom. IV. § 1, Pr. 10, Cor. 1.) This common intersection will therefore be the radial of all those original lines, and the intersection of this common radial with the plane of projection is the common vanishing point of those original lines.

This property of one vanishing point being common to any number of original lines, provided these are parallel, is the chief theorem on which practical linear perspective depends: and as all the phenomena of the modification of the *real* forms of objects, as seen by the eye, are referrible to it, by dwelling on it here, the learner will not only be more impressed with the principle, but will feel an additional interest in the subject by having the connexion of the geometrical truth with the art of painting pointed out to him.

If a long line of buildings, such as a street, be looked at, all the principal lines of which are parallels, and either horizontal or vertical, the former will *appear* to approach each other, or to *converge*; and if these horizontal lines are attentively considered, they will appear to approach the end of an indefinite imaginary line passing through the eye of the spectator, parallel to the originals. Because the rays proceeding to the eye from points in the originals in succession more and more distant, will make a smaller and smaller angle with such an imaginary parallel; and if the original lines of the buildings or street in question were sufficiently long, so as to extend beyond the limits of the power of sight, they would appear to meet in a point. If the spectator move to different distances from the buildings, this apparent point of convergence will seem to move with his eye, being determined by the imaginary parallel through it before mentioned, which is, in fact, the *radial* of the lines in question.

Now if the spectator conceive a transparent plane placed at any convenient distance before his eye, he will easily understand that this radial would cut this plane in a vanishing point; to which lines, drawn corresponding to those of the buildings as they appeared to him, would tend: and if, instead of supposing the plane set up before him, he were to substitute a sheet of drawing-paper, held flat on a board, and attempted to draw by his eye, as it is termed, the appearance of the buildings, he would find, that in order to give a correct representation of them, the lines in his drawing must converge to a vanishing point supposed to be found on the paper by such an imaginary radial.

If the spectator stand in the middle of a street, the horizontal lines of the buildings on both sides of him would *appear* to converge or *vanish* to the same point; this arises from these lines being parallel, the opposite buildings being generally so, and therefore both sets of lines in the two rows of houses having a common radial and vanishing point.

A street is mentioned, because it is there that the longest straight lines are to be found in nature, and therefore the apparent convergence, or *vanishing* of them, is most readily observed: but as the principle holds true whatever may be the direction or length of the original lines, provided they are parallel, the reader can satisfactorily account by the same reasoning for all the forms assumed by objects containing parallel right lines. Thus the *appearance* of the floor, walls, and ceiling of a room, the aisle of a cathedral, the sides and top of any rectangular box, or even a book, is referrible, in all these objects, to this cause, namely, the common vanishing point or *apparent* point of convergence of lines *really* parallel. These examples will be referred to again in illustration of the principles about to be explained, and the other phenomena of apparent form will be shown to depend on analogous causes.

10. A series of parallel original lines may have three relative positions with regard to the *plane of projection.*

They may be either parallel, perpendicular, or inclined to it, in any angle, as was before stated of a single line.

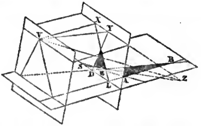
In the first case, none of them can have an intersecting point, as has been shown (4); and since their common *radial* will also be parallel to the plane of projection, it can never meet that plane, and consequently they can have no vanishing point.

11. On the second supposition, of the original parallels being perpendicular to the *plane of projection*, their common radial will also be perpendicular to it: therefore the *centre of the picture* (Def. 7.) will be the common vanishing point of all lines perpendicular to the *plane of projection*.

12. On the third supposition, of the original parallels being inclined to the *plane of projection* in any angle; it is obvious that their radial will cut that plane, somewhere, in an equal angle, and that such lines will have a vanishing point, but that this can never coincide with the centre of the picture.

13. It has been shown (8), that the perspective projection of any original line passes through its intersecting and vanishing points: consequently, when the former points of any number of original parallel lines have been found, lines drawn through these intersecting points and through the common vanishing point, will be the perspective projections of all those original lines.

14. Original lines parallel to ench other have as yet been alone considered; but now let any number* of lines, S E, D B, L A, &c., *not* parallel, be assumed as lying in one original plane. Let S, D, and L be the *intersecting points*, and let the lines cut each other in the points A, B, and E.



15. The radials of these original lines being respectively parallel to them, and all passing through the *vertex*, they must lie in one plane, which will be parallel to the original plane, and will pass through the vertex. (Geom. IV. § 2, Pr. 15.)

Def. 12. The plane passing through the vertex, parallel to any original plane, is called the *vanishing plane* of that original plane: and

Def. 13. The intersection of the *vanishing plane* with the plane of projection is called the *vanishing line*, and its intersection with the vertical plane is termed the *parallel of the vertex* to the original plane.

---

[*] Though only three are shown in the diagram, any number may be supposed, and the same reasoning will apply to them all; hence the references are expressed as if an indefinite number were alluded to.

16. Hence it follows, that the vanishing plane being a plane parallel to an original one, and passing through the vertex, any number of original parallel planes will have a common vanishing plane.

17. All original planes that are parallel will, hence, have a common *vanishing line.*

18. The vanishing points (Def. 10.) of all lines lying in the same, or in parallel original planes, will be points in the common vanishing line of those planes. Because the radials of all the original lines lie in one vanishing plane, and will therefore cut the plane of projection in the intersection of these two last-mentioned planes, which is that common vanishing line.

19. Hence, also, the point in which the vanishing lines of two planes cut each other, will be the vanishing point of the common intersection of those planes.

20. The vanishing line of a plane, or of many parallel planes, will be parallel to the *intersecting line*, or lines, of that plane or planes, and also to the *parallel of the vertex* and to their *station lines,* these four lines being the intersections of parallel planes, namely, the original planes and their vanishing plane, with the plane of projection and the vertical plane.

21. The *vanishing planes* of all planes perpendicular to the *plane of projection*, will be also perpendicular to that same plane; and the vanishing lines of such original planes will pass through the centre of the picture. And no plane that is not perpendicular to the plane of projection can have its vanishing line pass through the centre of the picture.

22. The vertical plane (Def. 3.) is the vanishing plane of all original planes which are parallel to the *plane of projection.*

23. Hence no plane parallel to the plane of projection can have any vanishing line, or parallel of the vertex. It has been already shown (1) that it can have no intersecting nor station line.

24. As the vanishing planes are always parallel to the original planes, it is also obvious that the vanishing planes of any original ones will form the same dihedral angle (Geom. IV. § 1. Def. 4.) that the original planes themselves do. (Geom. IV. § 2. Schol. to Prop. 17.)

25. As the radials of any original lines are parallel to those lines, and as the intersecting line and parallel of the vertex of the original plane are also parallels, it follows that the radials will make the same angles with each other and with the parallel of the vertex that the originals do with each other and with the intersecting line. (Geom. IV. § 2. Pr. 15.)

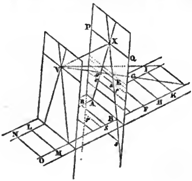
If the line L B (see the last figure) be perpendicular to the intersecting line S L, it will have its vanishing point X in the intersection of a radial V X through V, perpendicular to the vanishing line X Z, the angles V X Z and B L S being right angles; and therefore X will be the vanishing point of all lines perpendicular to the intersecting line.

Def. 14. The *centre* of any *vanishing line*, is the point where a radial from the vertex cuts it at right angles, and is the vanishing point of all lines perpendicular to the intersecting lines of the corresponding original planes, and drawn in them. This radial perpendicular to any vanishing line is called the *principal radial* of the original plane.

The vanishing point of any other original line, as D B, (last figure,) will lie nearer to, or farther from, the *centre* of the vanishing line, according as the original makes a greater or less angle with the intersecting line S L: thus the line S A makes a more acute angle with S L, than D B makes with S L; the corresponding radial V Z will consequently make a more acute angle with the vanishing line X Z than V Y does; therefore Z is farther from the centre X than Y is, and the radial V Z will be longer than the radial V Y, (Geom. 1. § 2. Pr. 12, Cors. 2 and 3.) In the same way it may be shown that the intersecting points of original lines are more and more distant from L, as those lines make less angles with the intersecting lines.

26. When the original lines are parallel to the *intersecting lines* of the planes in which they lie, and therefore parallel to each other, they will have neither intersecting nor vanishing points; or these points may be conceived at an infinite distance in opposite directions, and the perspective projections of all such original parallels will be parallel to the intersecting and vanishing lines of the planes in which they lie, or may be considered as passing through these infinitely distant points.

Let R S, E F, G H, &c., be any number of original parallel lines lying in a plane, and also parallel to the *intersecting line* A B; P Q being the vanishing line of the plane. The *perspective*



projections of these parallels will be parallels to the intersecting line A B, because these projections will be the intersections, of projecting planes passing through the original lines and the vertex, with the plane of projection. (Geom. IV. § 1. Prop. 10.)

Such of these original parallels E F, I K, &c., as lie *beyond* the plane of projection from the vertex V, will have their projections *e f*, &c., between the intersecting line A B and the vanishing line P Q; and the farther from A B the original lies, so much the nearer to the vanishing line will be its projection: hence this vanishing line may be considered as the projection of a parallel to A B at an infinite distance.

27. If the original parallels, as R S, &c., lie *between* the plane of projection and the vertical plane, their projections *r s* will lie on the farther side of the intersecting line A B from P Q; and the *station line* of the original plane, which may be considered one of the parallels, will have no projection, because its projecting plane coincides with the vertical plane, and is therefore parallel to the *plane of projection.*

28. If the original parallels L M, N O, &c., lie *beyond* the vertical plane from the plane of projection, their projections will lie above P Q from A B, and the farther the original is from A B, the nearer its projection will be to P Q; so that this *vanishing line* may be also considered as the projection of a parallel at an infinite distance from A B in a contrary direction to that, in which the former infinitely distant parallel was supposed to lie, of which P Q was also considered the projection. (See § 8.)

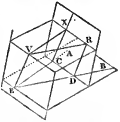
Let any two lines N I, O K be drawn parallel to each other in the plane of the parallels, so as to cut them all, and intercept equal segments of them, as N O, R S, G H, &c., &c. Then V X will be the radial and X the vanishing point of these two lines, V X being drawn in the vanishing plane (15) parallel to them; and X A, X B, drawn through their intersecting and vanishing points, will be their perspective projections. (8 and 13.)

It is obvious that the portions of the indefinite parallel projections of the original lines E F, I K, &c., which are intercepted between the projections X A, X B, are the *images*, or projections, of the corresponding *equal* segments of the originals; and that the farther from A B the original segment lies, the nearer to P Q, and consequently the shorter, is its projection, because it is intercepted nearer the point of convergence X, or vanishing point of the lines cutting off the original segment: and hence X itself may be regarded as the image of a segment at an infinite distance, intercepted by the two parallels infinitely prolonged from A B in either direction.

The radial of an original line is parallel, as has been shown, to that original, it being the intersection of the projecting plane with the vanishing plane (8, 9.); and the intersection of the same projecting plane with the vertical plane will be parallel to the perspective projection of the original line, being also the intersection of two parallel planes by the same projecting plane.

Def. 15. This intersection of the projecting plane of an original line with the vertical plane, is called the *vertical* of that original.

29. Consequently an original line, its *radial*, its *perspective projection*, and its *vertical*, always form a parallelogram, the radial being equal to the segment of the original lying between its *intersecting* and *station* points, and its *vertical* being equal to the segment of its projection, lying between its intersecting and vanishing points.



If two or more original lines A E, B E, meet in a point E, in the station line of the plane in which they lie, their perspective projections C X D R, will be equal and parallel to each other, because these are parallel and equal to the common *vertical* V E (Def. 15, 29); and the *vanishing points* X, R of such lines will be at the same distance from each other in the vanishing line, that the intersecting points C, D are from each other in the intersecting line.

It may assist the learner to recapitulate here the various definitions, for it is essential that he should remember them correctly.

Def. 1. The *vertex*, or *point of sight*, V, is the point in which the projecting lines, or *rays*, of original points, meet.

Def. 2. The *plane of projection*, or the *plane of the picture*, *a b d*, is that on which the image of an original point, line, or object, is produced, by the intersection of the rays from that point, line, or object, to the vertex, these rays being indefinitely extended in both directions.

Def. 3. The *vertical plane*, *f g h*, is one passing through the *vertex*, parallel to the plane of projection.

Def. 4. *Original planes*, *p m n*, are those in which any point T, line, or plane figure lie; all the planes bounding a plane geometrical solid, are original planes; and are always supposed to be indefinitely extended.

Def. 5. The *projecting plane*, *w y* D E, of an original straight line, T E, is one passing through the vertex, and in which the line wholly lies.

Def. 6. The *intersecting* and *station lines*, A D, B E, of an original plane, are its intersections, respectively, with the plane of projection, and with the vertical plane.

Def. 7. The *centre of the picture*, C, is the point in which a line, through the vertex, perpendicular to the plane of projection, meets this plane. The centre of the picture is, therefore, the point of the plane of projection, which is nearest the vertex. And the length of this perpendicular V C, between the two points, is the *distance of the vertex or of the picture.*

Def. 8. The *ray*, V T, of an original point T, is a straight line passing through it and the vertex.

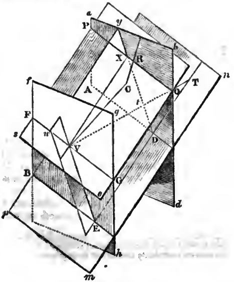
Def. 9. The *intersecting and station points*, D, E, of an original straight line, are the points in which it cuts the plane of projection, and the vertical plane, respectively; and are, in the intersecting and station lines of the plane in which the line lies.

Def. 10. The *vanishing point*, X, of an original straight line, is that in which a line through the vertex, parallel to the original, cuts the plane of projection.

Def. 11. The *radial*, V X, of an original line, is the line through the vertex, parallel to that original, by which the vanishing point is determined.

Def. 12. The *vanishing plane*, *s o* P Q, of an original plane, is one passing through the vertex, parallel to that original.

Def. 13. The *vanishing line*, P Q, of an original plane, is the common intersection of the vanishing plane of the original, with the plane of projection. The *parallel of the vertex*, F G, is the common intersection of the same vanishing plane with the vertical plane.



Def. 14. The *centre* R, *of a vanishing line*, P Q, is the point where a radial from the vertex cuts it at right angles; and this radial is called the *principal radial* of the original plane.

The principles which have just been explained, regarding the vanishing lines, planes, and points, of original planes and lines, admit of the most general and obvious elucidations from the appearances of natural objects.

Thus, to recur to the example before cited of the buildings in a street, the upright sides of the windows, which may be considered as parallel lines, *appear* to approach nearer and nearer to each other, the farther they are from the spectator; and the lines of the tops and bottoms of the same windows, which are parallel and horizontal, converge to the imaginary vanishing point before mentioned: the windows, therefore, appear to become smaller and smaller, till, if the street be sufficiently long, they look like short lines hardly perceivable, but bearing the same relative proportion to the whole apparent height of the house that those do which are close to the person looking at them: and if the street be imagined indefinitely prolonged, the vanishing point will be the representative of the infinitely distant window.

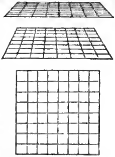
The joints of the stones of the flag pavement in a street, or a long room, as Westminster Hall, apparently diminish in distance as they recede from the spectator towards the farther end of the place; and he will find, upon attentively considering them, that they obey the laws above explained, in the form they present to the eye; although he is aware, or afterwards observes, that the stones constituting such a pavement are really equal squares or rectangles*.

It may be alleged, that in looking at a street, or any of the objects mentioned, there is no plane of projection to receive the images, and that therefore the connexion between the geometrical principles and the forms of the natural objects is not sufficiently clear. But it must be recollected, that the supposition of the image being produced on an interposed plane of projection, in no way alters the form of the pyramid of rays proceeding from an object to the eye of a spectator, and that it is the modification of this imaginary pyramid, by a relative change of situation of the eye and the object, that causes the change of apparent form of that object, because by such relative change of situation the image on the retina is really altered.

Thus let a chess-board be the object looked at, and let the place of the eye of the spectator be supposed to remain in the same point; the rays from the lines forming the squares on the board will form quadrilateral pyramids, of which all the bases are really equal and the vertex common; but the *form* of these pyramids will be very different according to the relative distance of the squares constituting their bases: the farther the square from the eye, or the farther from the vertex the base is, so much the smaller the solid angle (Geom. IV. § 1, Def. 7.) formed at the eye, or vertex, or so much the sharper-pointed the pyramid; the image on the retina will be so much the smaller, and consequently convey to the mind the idea or perception of a smaller form, though that form is really known *not* to be smaller, but to be equal to the others with which it is compared. If the chess-board be placed so that the eye is in a line perpendicular to the middle of the board, and at a considerable distance, compared to the size of it, the pyramids of rays will have nearly equal solid angles at the vertex, and the squares will be *seen* of nearly the same size and form, or as equal squares, whereas, when the board is held very obliquely, the squares nearest are not only in appearance larger in area, but different in form to those more distant, as will be easily recognised from the annexed figures.

---

* The reader cannot have a better object to exemplify these theorems, than a common chess-board, which he may place in different positions: from the regularity of the squares, he will perceive more easily the truth of these principles.



Let the same chess-board be supposed to be viewed from the same fixed point in the same position, and of course, therefore, presenting one constant appearance, or forming a constant image on the retina, as long as the eye and board retain the same relative situations. But if these constant pyramids of rays be supposed to be cut by a plane in different angles and positions, the perspective projections, on this plane in these different positions, will be very different, or form very different figures, or *images*, of the original squares.

Thus if the cutting plane be supposed parallel to the plane of the board, or to the bases of the pyramids of rays, the images, or projections, of the squares will be squares equal and similar to each other, though smaller or larger than the original squares of the board, accordingly as the *plane of projection* was nearer or farther from the vertex than the board itself. (Geom. IV. Prop. 15. Cor.: demonstration to Prop. 30., and Schol. to Prop. 29.)

If the cutting plane, or *plane of projection*, were supposed *not* parallel to the board, the lines of the board which form the real squares will have *vanishing points*, and the *images* of the squares will be no longer squares, but trapeziums, or other quadrilateral figures, according to the relative positions of the intersecting line of the plane of the board with the plane of projection, as has been explained in the preceding propositions. And if the plane of projection move parallel to itself in the same angle of obliquity to the board, the figures of the squares on it will remain *similar*, but diminish or increase, in area or size, as the cutting plane approaches to, or recedes from the vertex.

Hence any number of different images of the same forms may be obtained, while the real appearance of the squares remains unaltered to the eye, kept in the same point; and yet these very different images are all correct perspective projections of the same object.

The explanation of this apparent contradiction is derived from the correct conception of what the perspective projection of an object is, and that such a projection will only give a correct image of the original, when the eye is placed in the same point, with regard to the plane of projection on which the image is formed, as it must be to cause that image to be a true section of the pyramid of rays proceeding from the object to the eye, for in that situation only will the rays from the image correspond with those from the object itself, and consequently produce a similar image on the retina.

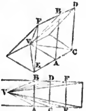
If the eye of the spectator and the chess-board keep their same relative position, and any number of planes be interposed between, or be placed beyond the board and eye, the images which would be formed on them, by projecting planes passing through the lines of the original squares, though absolutely very different on the different planes, will still coincide, when viewed from the same point, with the original lines; and if these figures were correctly coloured, they would be all equally correct pictures of the chess-board, as seen from that point; but if any of them be looked at from any other point, the representation will no longer be true, or will not produce an image on the retina the same as that which the original board would produce.

It has been shown (§ 26, 27, pp. 132, 133.) that the indefinite projections of straight lines, parallel to the plane of projection and to themselves, will be straight lines, parallel to the intersecting and vanishing lines, and therefore to one another; because the projecting planes of such original lines will be cut in parallel straight lines by the *plane of projection*. Now let anybody, convinced of this geometrical theorem, stand before a long wall, the top and bottom of which may be considered as parallel right lines; these, instead of appearing *parallel*, will seem to converge or approach each other in opposite directions on each side of him: as he is aware that no right lines could do this without forming an angle, and as he not only perceives none, but is convinced that none really exists, in the top and bottom lines of the wall, he is persuaded at last that the outlines are curves, and not straight lines.

and still less parallel lines: here, then, is an apparent inconsistency between the geometrical principles and the real appearance of objects. As the explanation of this difficulty will throw a still clearer light on the connexion between linear perspective and its application to drawing, it will be entered on at length.

To simplify the case, let the top and bottom of the wall be supposed two geometrical right lines F D, E C, truly parallel to each other, and let parallel lines or ordinates, perpendicular to both, be supposed drawn on the face of the wall at equal distances, thus dividing the surface of it into equal rectangular spaces.

Now the rays from the two horizontal lines will lie in two planes, the intersection of which will be a right line parallel to them (Geom. IV. § 1. Prop. 10. Cor. 1.), and passing through the vertex V or place of the eye. Let the rays from the extremities of one of the perpendicular ordinates, A B, be at right angles to this intersection; then the angle B V A contained by these two rays will be greater than the angles D V C, F V E contained by the rays from the extremities of the ordinates C D, E F lying one on each side of the former, and these angles will decrease as these ordinates recede farther and farther on each side from the first A B; because each pair of rays and the ordinate from which they proceed form triangles, the bases of which, or the ordinates, are equal, and the sides, or rays, constantly increasing in length: therefore the angle subtended by the equal bases will diminish, as is plainly seen from the lower figure.



Now, it is the magnitude of the angle subtended at the vertex, or eye, that determines the image of the ordinates made on the retina; and the farther the ordinates recede on each side from the central ordinate A B, the smaller this angle is shown to become; and consequently the apparent outline of the horizontal lines of the wall, which is determined by the length of these apparently diminishing ordinates, will appear to converge on each side of the spectator.

If a plane be supposed to cut the rays from such a wall, parallel to the plane of it, the section of the two planes passing through the horizontal lines of the wall will be two parallel right lines, as before mentioned, and the images or projections of the perpendicular ordinates will be parallel and equal lines at equal distances from each other, because the quadrilateral pyramids of rays are supposed to be cut by this plane parallel to their bases, and therefore the sections will be similar to those bases; but these being equal rectangles, their images, or these sections, will be equal rectangles. When viewed from the right point, these equal images will subtend less and less angles, as they lie farther and farther from the vertex, or eye, on each side, on the same principle as above explained of the real ordinates on the wall; and consequently their apparent diminishing, in length, will give to the right lines of the projection of the top and bottom of the wall, the same curve that the real outlines assume to the eye, and such an image or picture will not give a correct idea of the original object unless viewed from the right point or from the true vertex.

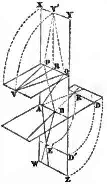
Another explanation of an analogous discordance between the theory and the real appearance of objects, will not be misplaced here. If a spectator stand opposite the middle of a colonnade, the column nearest him, or that to which he is opposite, will obviously *appear*, not only the highest, but the thickest in diameter, or altogether the largest, that is, it will form the largest image on the retina; yet if the rays from the columns were cut by a plane parallel to that passing through their axes, all the *images* of the columns on it would be of the same height, for the reasons above given, accounting for the parallel lines representing the top and bottom of the wall: but the *diameters* of the images of the columns would *increase*, the farther the originals receded on each side from the spectator; because the two planes of rays, tangential to the two sides of each more distant shaft, would be cut more obliquely, and consequently, the intercepted segment, which constitutes the *diameter* of the image, would be greater and greater.

Such a perspective projection, or view, of a colonnade, however correct in theory, would be revolting to the eye, as directly contradicting the evidence of the senses; consequently no judicious artist ever attempts to delineate a long wall or colonnade as seen from such a point of view, because he must either falsify the accuracy of his outline, which will lead him into errors in other related parts of his design, or, if he observes the geometrical principles, his drawing will be preposterous, since it is very seldom that any design is viewed from the precise, and only point, which would render it a correct image of the original object.

§ OF THE PERSPECTIVE PROJECTION OF RIGHT LINE FIGURES, &c.

Having thus shown the general relations between original planes and lines, and their vanishing planes, lines, and points; by means of which the *perspective projections* of the former are indefinitely determined, it will be necessary to explain the mode in which these principles are practically applied, before entering on the subject of the definite images of finite lines, composing figures or solids.

Any original, vanishing, or projecting plane, may be conceived to be turned round on its intersection with the plane of projection, or with the vertical plane, till it entirely coincides with these; the paper or drawing board may then represent either, or both, of these last mentioned planes, with the others brought to coiucide with them, according to circumstances; and any constructions can be practically made on it.



Let V be the vertex, A E, B E two original lines lying in an original plane, of which A B is the intersecting, P Q the vanishing lines: and V P, V Q the radials of the original lines; now if the original and vanishing planes be supposed turned round on A B and P Q, till they coincide with W X Y Z, the *plane of projection*, by this supposition the vertex V will be brought to V′, and the radials will be V′ P, V′ Q, which will obviously form the same angle with each other that they did in their original position; the triangles V P Q, and V′ P Q being equal in every respect.

In the same way, if the original plane be turned round on A B, till it coincide with W X Y Z, the original lines in it will fall into the situation A E′ B E′, and will form the same angles with each other as before; and any other original line as E D, when brought into the plane of projection by this supposed revolution on A B, will bear the same relation, in every respect, to the intersecting and vanishing lines of its plane, that it did when in its first position. The radials V′ P, V′ Q will also preserve their parallelism to B E′ A E′ when the planes are brought to coincide: and as it is very important that this should be clearly understood, from the facility it affords to practical perspective, it will be demonstrated here at length.

Let V R be perpendicular to P Q; then V R, during the revolution of the vanishing plane, will describe a plane which will be perpendicular both to this last, and to the plane of projection; consequently R V′, also perpendicular to P Q, will be the intersection of this plane formed by V R, with the plane of projection, (Geom. IV. § 1. Prop. 3. Cor. 1.) and as V R and P R remain of the same length, the triangles V P R, V′ P R are similar and equal in all respects, therefore V′ P is equal to V P[*]: the same demonstration will apply to the triangles V R Q, V′ R Q; and, therefore, V′ Q is equal to V Q, consequently the whole triangle V′ P Q has its three sides equal to those of V P Q, and is, therefore, equal and similar to it in every respect.

By precisely the same mode of demonstration, the triangle A E′ B, formed by any two original lines, and the intersecting line of their plane, when brought to coincide with the plane of projection, may be shown to be equal and similar to the same triangle A E B, when in its original position. Now the radial V P being parallel to B E, (Def. 11.) and P Q being also parallel to A B, (20.) the angles V P Q, E B A are equal, (Geom. IV. § 2. Prop. 15.) and since it has been just shown that the angle V′ P Q is equal to V P Q, and the angle E′ B A to the angle E B A, therefore V′ P Q, E′ B A are also equal, and as P Q, A B are parallel as before, V′ P is also parallel to E′ B. (Geom. I. § 3. Prop. 18.)

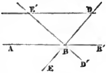
30. The same demonstration will of course apply to any other original lines, and their radials; and consequently the radial of an original line will always be parallel to it, when both are brought into one plane, as that of projection; or when into any other position than the original one; provided the two planes in which they lie are supposed to preserve their parallelism.

31. Hence the whole surface of the drawing board may represent, either the plane of projection, or any original plane brought to coincide with it, by being made to turn on its *intersecting line*; or any *vanishing plane* turned round on the *vanishing line*, till it coincide with the same plane of projection; and since all planes are supposed indefinitely extended, the space on each side of these lines on the drawing board may represent part of these respective planes; great attention must, therefore, be paid, to ascertain in practice which plane is intended during any part of the construction; and what parts of an original line, with respect to the

---

[*] It will be easily perceived that P R is the axis of a right cone formed by the revolution of P V round a circle, of which R V is the radius; this remark will apply whenever one plane revolves on its intersection with another; and the reader will often have his conception of the effect produced by the different planes in perspective problems materially facilitated by considering them in this manner.

vertex and plane of projection, the two segments on the drawing board on each side of the line on which the plane is supposed to be turned, represent. Thus if A B represent on the drawing board the intersecting line of an original plane, the segment D B represents the part of an original line *between* the vertical, and the plane of projection, and E B the part *beyond* the plane, supposing the original plane to be turned round in one direction; but if this plane be supposed to be turned round in the *contrary* direction, the *same* original line will assume a new aspect on the board, and D′ B will be the part between the vertical and plane of projection, while E′ B now represents the part more remote. In all practical applications of perspective, these complicated relations of lines occur, and patient attention and thought will alone enable the student to avoid the errors he may commit by not rightly understanding them.



Let a line A B (see fig. p. 145.) be given* or assumed at pleasure, on the drawing board, as the *intersecting line* of an original plane making any angle with the plane of projection. Let C be the centre of the picture, and let D E, D F, E F, be three original lines in the given plane (A B)† forming a triangle, the side, or line, D E, being parallel to A B.

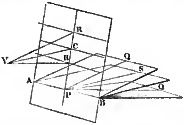
32. To obtain the vanishing line of the original plane; draw C V from C the centre of the picture parallel to A B, and equal to the distance of the vertex. (Def. 7.) Make the angle C V R equal to the complement of the angle which the original plane makes with the plane of projection, and draw C R perpendicular to A B, cutting V R in R; a line through R, parallel to A B, will be the *vanishing line* of the original plane.

On which side of C V the angle C V R is to be constructed, depends on the direction in which the original plane is inclined to the plane of projection. Let C be the centre of the picture, V the vertex, and A B the intersecting line of the given plane; let a plane passing through the vertex, perpendicular to the plane of projection, and to the given plane, cut the former in R P, which will pass through C, because V C is perpendicular to the plane of projection, (Geom. IV. § 2. Pr. 18.) and let it cut the original plane in P Q, which will be perpendicular to A B; then the angle R P Q is the angle of inclination of the original plane to the plane of projection; and if this angle is acute, the principal radial, V R, (Def. 14.) will cut the plane of projection on the other side of C from A B; but if the angle R P Q is obtuse, then V R will meet the plane of projection nearer to A B than C. The angle C V R is obviously equal to the complement of R P Q, for if another plane A B S passed through A B, perpendicular to the plane of projection, this latter would form, with the first original plane, an angle, which would be the complement to a right angle of R P Q, and would be measured by the angle Q P S, to which R V C is equal; the radials V R, V C being parallel to P Q, P S. (Geom. IV. § 2. Pr. 15.)

---

* Any original planes, lines, or points, may be assumed as given and at any angles with each other, as may also the centre of the picture, and the distance of the vertex, or the length of the radial from the vertex perpendicular to the plane of projection. (Def. 7.) When the rules of perspective are applied to delineate real objects, such as buildings, machines, &c., the original planes of which are consequently given, the relative position of these, with regard to the plane of projection, is determined from other considerations.

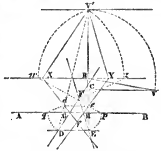
† To avoid unnecessary repetition, an original plane will be expressed by its intersecting line, vanishing line, or by its intersections with other planes, the context always immediately showing which plane is meant: thus in the text, (A B) will express the given original plane, the letters being placed between brackets when the plane, and not the line itself, is referred to.



Now let the triangle R V C be turned round on R C, till it coincides with the plane of projection; the angle R C V being a right angle, and remaining so, unchanged by the revolution of the triangle, C V will become parallel to A B, and the angle C V R (see following figure) being constructed on the drawing board, equal to the complement of the angle of the original plane, (A B) to the plane of projection, as directed in the above construction, V R will give R the centre of the vanishing line of A B. (Def. 14.)

If the given plane (A B) were supposed perpendicular to the plane of projection; then its vanishing line must be drawn at once through C, the centre of the picture, parallel to its *intersecting line*, for in this case V R and V C would coincide. (21.)

Having thus obtained the vanishing line of the given plane (A B), draw R V′ perpendicular to it, and equal to R V, its principal radial (Def. 14.); through V′ draw a line, parallel to A B and the vanishing line; this line will be the *parallel of the vertex*, (Def. 13.) brought into the plane of projection. Draw the radials, V′ X, V′ Y, parallel to the original lines D F,

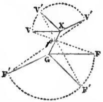


E F, or, making the same angles with each other, and with the parallel of the vertex, that those originals do with each other, and with the intersecting line A B (25); then X, Y, where these radials meet the vanishing line, will be the *vanishing points* of D F, E F respectively (18); and lines drawn through these vanishing points, and through the intersecting points G, H, will be the *projections* of the originals; the point *f*, where they cut each other, will be the projection of F, the point of intersection of the originals (8).

The point F, from the situation of the original lines, with respect to the intersecting line A B, obviously lies between this last line, and the station line; consequently its projection *f* will be on the other side of the line A B from its vanishing line (8, 31).

If a line be drawn from V' to F, it will pass through *f*: such a line will represent the ray by which the image *f*, of F, is determined; this ray would of course pass through *f*, if the different planes were in their original relative situations to each other; but that the line F V' will also pass through *f*, when these various planes are turned round so as to coincide in one, is proved by the following theorem.

33. If V X, G F, be two finite lines, drawn parallel to each other from the extremities of a line G X, and a line V F be drawn, cutting G X in *f*; then, however the lines X V', G F' be supposed turned round on X and G as centres; provided they are always parallel, the line V' F' joining their extremities will cut the line G X in the same point *f*.



For the revolving sides, always being supposed parallel, and of a constant length, they form similar triangles with the line V F, joining their extremities, and with the segments of the constant line G X. This line G X is therefore always cut in the point *f*, into segments, having the same ratio, namely that of the revolving sides. (Geom. II. Prop. 31.) But a line of given length, always divided in a given ratio, will always have the same segments; therefore *f* is a fixed point.

Hence if X be the *vanishing* and G the *intersecting* point of a line G F, V X its radial, and V F the ray; this last will cut the projection G X in the same point *f*, when the radial V X, and the original line G F, are turned round, so as to come into one plane, viz., the plane of projection, provided the line and its radial are still parallel to each other, and of the same lengths respectively that they originally were.

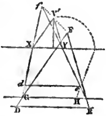
This principle is of great and constant use in practice; for if the image *f*, of a point F, be not accurately enough determined by the intersection of the projections of two lines, in which F is situated; it is only necessary to set the length of the radial along *any line* drawn from the vanishing point, and the length of the segment of the original line, from its intersecting point to the point F, along a line drawn parallel to the former from that intersecting point; then a line joining the extremities of these parallels will cut the projection of the line in the image *f*, of F, more distinctly, provided the parallels are drawn in the direction best calculated for that purpose.

34. And since, in the above demonstration, the consequents would remain the same if the antecedents were multiplied or divided by any the same number; in the practical construction any the same multiple, or part, of the radial and original segment, may be made use of to determine the projection *f* of the original point; this is frequently done when the lines are too long to admit of their being set off on the drawing board.

35. The projections of the points D, E (figure p. 145) are best found by means of the principle just explained, for the line D E being parallel to A B, and therefore having no intersecting nor vanishing points (10), the projection of it cannot be obtained by the usual method. Set the length of the radial V′ Y, from Y, along the vanishing line to *w*, and the length of the segment H E of the original line F E from its intersecting point H, along the line A B to *p*, but in the contrary direction; then the line *p w* will cut *f* Y in *e*: *d* might be found by a similar construction, made with the radial and segment of F D, but since the projection of D E must be parallel to A B, because the original is so, it is sufficient to find one point *e* in that projection; then a line drawn through *e* parallel to A B will be the projection of D E as required.

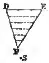
The rays V D, V E might have been made use of to obtain *d*, *e*, but they would have cut the projections *f* X, *f* Y too obliquely to allow of the points *d*, *e* being accurately determined.

On comparing the triangle *d e f*, or the projection of D E F, with its original, the effect of viewing a right line figure obliquely will be apparent: whatever may be the distance of the vertex (or place of the eye) from the plane of projection, if the plane of the original figure be *not* parallel to this latter, the *perspective projection* of the original will be unlike it, and may even be totally different: thus if the point F were in the station line, that is, if the perpendicular distance of F from A B were equal to the length of the principal radial V R, then the original lines F D, F E would be projected into parallel lines (28, 29), for in that case F could have no projection, and an original triangle would be projected into three lines which would not form a triangle.



36. If the point F were farther from A B than V is from X Y, G H would be longer than X Y, and the projections of D F, E F drawn through G and X, H and Y instead of intersecting below A B, as in the first case, or instead of being parallel as in the case last supposed, would meet *above* the vanishing line in the point *f*, which would be the projection of F, and the whole original triangle would be represented by a triangle *d e f*.

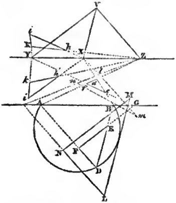
37. It is almost needless to remark, that this last supposition arises from the purely geometrical manner in which the subject of perspective is treated, by dropping all consideration of vision, and therefore cannot be realized in nature, since it is of course impossible for the eye to see at the same time the two parts of the same figure, one lying before it, and the other behind it; but in practical linear perspective, the projection of an original point situated, as it were, behind the vertex, is of common occurrence, when the principles are applied to interior views, or to the projection of shadows.



38. The last two hypothetical cases of the projection of the converging lines into parallel lines, or into lines intersecting above the vanishing line, will admit of a certain degree of exemplification by real objects. If a large plane triangle D E F be chalked, or marked in any way, on the ground, and the spectator look at it attentively from any point S out of it, he will see it as a triangle, but unlike the original, the acute angle D E F would probably appear obtuse; and the other two, much more acute than they really are, according to the height of his eye above the floor, and his distance from the nearest angle F: but if he place himself on the point F, that point will of course be unseen by him as long as he keeps his eye steady; and since imaginary parallel ordinates between the two sides F D, F E, would subtend proportionably greater angles at his eye, the nearer they were to his foot, or to the angle F, the lines D F, E F, instead of appearing to approach each other as they really do, would *appear* to be parallel. If he stood within the triangle, these sides would still less appear to converge *towards* him, that is, they would appear to converge *from* him; for it is clear that if the triangle were sufficiently large, the side D E might

be so distant as to appear very much smaller than any line, supposed to be drawn on the floor parallel to D E, between the other two sides, but near to the point where the spectator stood; yet, since D E will always subtend some angle, unless it were infinitely distant, D E will always appear as a line, and form one side of the triangle, and the other two will appear to approach each other, the farther they were extended from him.

39. Since lines drawn parallel to the sides of any triangle will form triangles similar to the first; if the vanishing points X, Y, Z of the three sides of a triangle be given or found, any lines drawn through X, Y and Z will be the projections of parallel lines, and will consequently form the projections of triangles, as *h i k, l m n*, *similar* to the original of *d e f*, the first triangle.

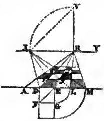


Let *d e f* be given as the projection of a triangle, Y Z as the vanishing line, and A G as the intersecting line of its plane: if the situation and distance of the vertex be unknown, the original triangle cannot be determined, since *d e f* may be the projection of any number of different triangles, to different points taken as vertices. To determine that original, therefore, one other datum must be assumed. Let the original of the point *d*, and angle *f d e* be given or assumed.

Produce Z *d*, Y *e*, X *e* to the intersecting points A, G, and B of the original sides; on A B describe the segment of a circle which shall contain the assumed, or given, original of the angle *f d e*. (Prob. 39, p. 69.) From any given or assumed point D, in the arc, draw D A, D B, and draw lines from X and Y parallel to D A, D B, meeting in V: join V Y: through G draw a line parallel to V Y, cutting D A in F, and D B in E: then the triangle D E F will be the original of which *d e f* is the projection, V being the place of vertex, and *h i k*, *l m n* will be the projections of triangles similar to D E F, which triangles may be found by drawing lines parallel to the sides of this last, or parallel to the radials V X, V Y, V Z, through the intersecting points, as is shown in the figure, of *l m n*, the original of which is L M N.

40. It is obvious that the point D might be taken in the segment on A B, so as to fulfil some other condition relative to the original triangle D E F, as well as that of having an angle F D E of a given magnitude, or, if instead of deriving the place of the vertex from the assumed position of one point D of the original of a given image *d e f*, V were assumed first; then the points D, E and F would be determined by drawing lines from A, B and G parallel to the radials V Z, V X, and V Y; but in either case, *h i k*, *l m n* will be the projections of triangles similar to D E F.

41. A similar converse of the perspective problem, or the finding the original figure, of which a given one shall be the projection, is subsequently shown of a parallelogram, since it occasionally occurs in practice. (See p. 155.)



42. Let A B, X Y be the intersecting and vanishing lines as before, and D E F G be a square, the side D E coinciding with A B. And since, consequently, D F, E G are perpendicular to A B, the projections of these sides will have R, the centre of the vanishing line, for their vanishing point, and V R for their radial. (25, Def. 14.)

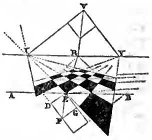
Make R X equal to R V, then X will be the vanishing point, and V X the radial of the diagonal of the square: for R V X will be an angle of 45°; E X, being therefore drawn, will be the indefinite projection of the diagonal E F, and a line through *f* parallel to A B will complete the projection of the square, since D E and its projections are identical (26). If another square, equal to D E F G, were supposed to be situated next to it, with its sides in the same lines, D F, E G produced, its image or projection is immediately obtained, by drawing *g* X, cutting D R in *f'*, and *f' g'* being drawn parallel to A B, will be the projection of the farther side from the vertex of this second square. Again, if *g'* X were drawn, another image of another equal square would be obtained, for the two sides of the originals being in the same lines as D F, E G; D R, E R will be their common projections, and since the diagonals of such a series of original squares thus situated would be parallel, X will be the common vanishing point of them all.

If E L, L M, &c., be made equal to D E, then L R, M R, &c.,

being drawn, they will be the projections of the sides of squares equal to the first, and L X, M X, &c., will be the diagonals. And the projection of an original consisting of equal squares will be produced, such as the squares of a chess-board or a marble pavement.

These images of the same or equal squares, it will be now seen, are very different, according to the situation of the respective original square with reference to the vertex, and these images become smaller and smaller as those originals are farther and farther from that vertex. (See observ. pp. 137, 138.)

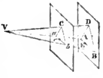
In the annexed figure, the squares are represented placed obliquely to the intersecting line; while this, the vanishing line and distance of the vertex remain unaltered, the projections are obtained by the vanishing points X, Y, of the sides, and the vanishing point R of the diagonals. Here the same or equal squares are projected into very unequal and dissimilar trapeziums, which are all, nevertheless, correct images of the same equal squares, as these would appear to an eye placed at the perpendicular distance of V from the plane of projection, or from the paper which represents it, the original squares being in the respective positions indicated by the squares D E F G, &c., &c.



Since the figure consisting of the original squares may be supposed indefinitely extended in every direction by the application of new squares; some of these would lie behind the station line, and consequently would have their images above the vanishing line X Y; the projections of such squares will be obtained by producing the lines E X, E Y, &c., and the trapeziums thus produced will be still more unlike the original, yet are nevertheless correct projections, or correct sections of the pyramids of rays proceeding from the points of equal squares to a common vertex at V, by a plane of projection intersecting the plane of those squares in A B.

The projections of right lines have hitherto been considered without any reference to their relative magnitude to the originals: but several principles of practical utility are deducible from these relations, which renders it necessary, before proceeding farther, to enter into some explanation of them.

43. Let A B be any finite right line, situated in a plane, parallel to the plane of projection; V the *vertex*, and V C the



distance of the vertex (Def. 7.), and let V C produced meet the original plane in D: V D will be also perpendicular to this plane. (Geom. IV. § 2, Pr. 11.) Then *a b*, the projection of A B, will be in the same proportion to AB, that VC is to V D; whatever may be their relative lengths; or however the plane of projection may be situated with respect to the vertex and original plane.

For draw *a* C, *b* C, and AD, BD; then the planes being parallel, the sides of the triangles *a b* C, A B D, will be respectively parallel, or the triangles will be similar, and the triangles V*a* C, V A D, will be similar, as will also the triangles V *a b*, V A B, therefore,
A B : *a b* : : V A : V *a*, and V D : V C : : V A : V *a*, consequently A B : *a b* : : V D : V C.

44. If the original line were divided into two or more segments in any proportion to each other, or to the whole line, by any point or points as at E, the projection *e*, of E, will divide the image *a b* in the same proportion as the original is divided by E (Geom. II. § 4. Pr. 30.), for each segment of the image is to the corresponding segment of the original line as V C to V D.

45. And since the lengths of V C, V D will remain always the same, wherever the vertex may be situated in the vertical plane (Def. 3.), as long as the three planes remain unaltered in their relative distances, the projection *a b* will remain of the same length, wherever V may be in the vertical plane; though the position of *a b* in the plane of projection will of course vary as V varies.

From these theorems the reason is obvious why the projection of a right line figure, the plane of which is parallel to the plane of projection, is always similar to the original; and from the last it appears that that projection will be of the same magnitude, wherever the vertex may be situated in the same vertical plane; because on this supposition the sides of the projection will be of the same length in every position of V; and they will always be in the same proportion to the sides of the original figure that the distance of the vertical plane from the plane of projection, bears to the distance of the vertical plane from that of the figure.

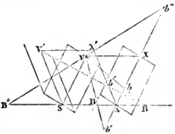
46. If the plane of projection lie *between* the vertex and the original, the projection of a right line, or of a right line figure, will be *less* than the original; V C being less than V D. If the plane of projection were *beyond* the original, or this be supposed between it and the vertex, then the projection will be *greater* than the original, for in this case V C would be greater than V D. And if V C were equal to V D, the image and the

original would be equal; this can only happen when the original plane either coincides with the plane of projection, or when the vertical plane is at equal distances between both.

47. But it is also clear that, in the first two cases, that is supposing the plane of projection and the original plane to lie on the same side of the vertex, that the farther from the former the vertical plane is, the more nearly will the projection approach in magnitude to the original; for in this case the more nearly will the ratio of V C to V D approximate to a ratio of equality: and if V C were infinite, or the vertex were at an infinite distance from the plane of projection, then V C and V D would be equal, both being infinite, and consequently the projection of the line or figure would be equal to the original.

This theorem will be hereafter, alluded to and exemplified when *orthographic projection* is explained.

48. Let A, S, be the intersecting and station points of an original line A S: V X its radial (Def. II.); X its vanishing point, *b* the projection of a point B situated any where in A S. Then the distance of *b* from X will be to the distance of *b* from A, in the same ratio that X V is to A B.

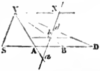


For V X being parallel to A S, the triangles X V *b*, A B *b* formed by these lines and the ray from B are always similar, (Geom. I. Pr. 15.) (Geom. II. § 14. Pr. 31) and consequently *b* X : *b* A : : VX : B A, whether the original point be at B, B′ or B″.

Hence, if in a construction, an original point in a line lies so far from its *intersecting point*, that it does not fall within the limits of the paper or drawing board, so that the ray cannot be drawn from it to the vertex, by which its projection can be determined; its image *b* may be found by dividing A X, either internally or externally, in the ratio of the radial V X of the original line, to B A, the distance of the given point from the intersecting point of the line, this latter being known.

49. Since the truth of the last proposition is totally independent of the angle at which the original plane is inclined to the plane of projection and vertical plane; it follows that, as long as the radial V X, the vertical V S, and the original point B, remain the same, the projection A X of the original line will be cut in the same point *b* by the ray from B, however the angle B A X or B S V may be varied by the planes turning round on the intersecting and station lines, provided they are always parallel;

thus if they were revolved till S V, A X come into the position S V′, A X′, the point b' in A X′ would be at the same distance from the intersecting and vanishing points that *b* was; because V X and B A remain unchanged.



50. If an original finite line A D be bisected by a point B, the projection of the line, included between the vanishing point X and *a*, the farthest image of one extremity of the line, will be harmonically divided by the images *b*, *d* of the other two points; that is *a b* : *b d* :: *a* X : *d* X. For the three rays from the points, and the radial of the line, form harmonical lines (Geom. II. § 6.), by which the projection *a* X of the line is therefore harmonically divided (Geom. II. § 6. Pr. 49.) This property affords a practical rule of frequent application; for the image *a b* of a finite line may be *perspectively** bisected by dividing the distance between the extremity *a*, and the vanishing point harmonically, in a fourth point *b*, (Prob. 15. p. 37.); the point *b* so formed will be the projection of a point B which bisects the original line.

And so also, if the image of a finite line as *a b* or *b d* were given, another point *d* or *a* may be found, which will be the projection of a point D, or A, cutting off another segment of the original line equal to the former.

Hence the images *a*, *b*, *d*, *e*, *f*, &c., &c., of points in an original line which is divided by those originals into equal parts; form an harmonical progression with the whole indefinite image of the line to its vanishing point; that is if *a b*, *a d*, *a* X; *b d*, *b e*, *b* X; *d e*, *d f*, *d* X, &c., &c., be harmonical progressionals, the original points will be at equal distances from each other in the original line.

51. And since the vertical of any original line (Def. 15,) is parallel, to its projection (28); if any segment of that projection, as *a d*, is geometrically bisected by a point *b*, the original line between its station point S and the further original point of the given segment, as D, will be harmonically divided by the originals of the other points, that is S D is harmonically divided in A and B.

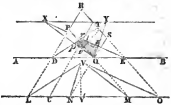
This is obvious from the preceding demonstration.

52. The same mode of proceeding, by which the projections of the square were obtained in the last example, is applicable to any parallelogram; but for the sake of illustrating the practical use of some of the foregoing principles, another example of such a figure will be given.

---

* The image of a finite line is said to be *perspectively* bisected by a point, when the original of that point *geometrically* bisects the original line; when this species of bisection is not meant, the bisection of an image of a line is called a geometrical bisection if the two parts are equal to each other.

The original plane of the figure P Q R S is employed in this case instead of the plane of projection; A B being the intersecting line, and V′ the point where the perpendicular from the vertex meets the station line L O; V′ V being the distance of the vertex from the station line, laid down on the original plane. Produce P Q, R S, R P, and S Q, to their *station points* M, O, L and N, and draw V L, V M, V N, V O: these lines will be the verticals (Def. 15.) of the original lines, and will consequently be parallel to the projections of those lines (28); if, therefore, lines be drawn through the intersecting points D, E, and Q, parallel to V L, V O, V M and V N, they will be the projections; and the trapezium *p* Q *r s* formed by them will be the projection of the original figure; the plane of projection being supposed turned down on the original plane.



Since A B and L O are parallel, D Q is equal to L N, and Q E is equal to M O, consequently the triangles D Y Q, L V N will be equal and similar, as will also be the triangles Q X E, M V O; in consequence, therefore, of the common vertex V of the two triangles, the vertices X, Y, of the two which are equal to M V O, L V N, must be in a line parallel to A B, and at the same distance from it that V is from L V; (Geom. I. Prop. 27.) Y X is therefore the vanishing line of the original plane, and these points, the vanishing points of the parallel sides of the parallelogram; and the projection *p* Q *r s* obtained by this construction is the same figure, as if it had been found at once by means of those vanishing points.

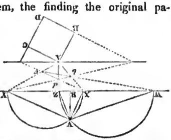
Let P Q, R S be bisected by the line T *v*, parallel to P R, then the projection of T *v* may be found by dividing the image Q X harmonically in the point *v* (Prob. 15. p. 37.) and *v* Y being joined, it will give the line *t v*.

It has been shown (49) that the points *s*, *t* and *r*, in E X, and the points *p* and *v* in Q X, will always be the same, whatever angle the plane of projection and vertical plane may make with the original plane, provided the *intersecting* and *vanishing lines*, the *station line* and *parallel* of the *vertex*, remain the same; and as this applies equally to the projections of all the points of an original figure, it follows that the projection of that figure will be the same, under the same conditions. The truth of this principle is also confirmed by the preceding construction, where no alteration of the angle of the planes would have affected the result, that angle not being in any way employed.

This might appear to contradict what was before remarked (p.138) on the circumstance of very different figures being produced by the sections of the same pyramids of rays, supposing the original and the vertex to remain constant, and the plane of projection to form various angles with the plane of that original; but it must be recollected, that if the original plane and vertex are supposed to preserve the same relative situations, all changes of position of the plane of projection, and consequently of the vertical plane *, will produce very different *intersecting* and *station lines* with respect to that vertex, and that, consequently, the lengths of the radials and verticals of the original lines will vary, as will also the distances between the intersecting points of those lines and the points of the original figure: while in the present supposition, all these lines and distances remain the same; and by the change of the angle of inclination of the plane of projection and vertical plane to the original plane, the relative distances of the vertex and the original figure are alone altered, this being precisely the converse of the former conditions.

This practical construction of the projection of an original figure, by means of the intersecting points and *verticals*, is not so often used, as that by means of the intersecting and vanishing points; because a line cannot be so accurately drawn from a point, parallel to another, or making the same angle as that other does with a third line, as it can be drawn through two points; but there are occasions on which it may be successfully employed in combination with the more usual method.

The converse of this problem, the finding the original parallelogram of which a given quadrilateral is the projection, is frequently required.



Let A *b c d* be given, to find the situation and distance of the vertex, so that A *b c d* shall be the projection of a square. Produce the opposite sides till they meet in the points X and Y, join X Y, draw and produce the diagonals of the figure to cut X Y in W and Z. On X Y and W Z describe semicircles cutting each other in V, then V will be the vertex, and V R perpendicular to W Y, the principal radial of the original plane. If a line be drawn through A parallel to the vanishing line, it will be the intersecting line of the plane, and the length of the side of the square may be found on it, by setting either radial V X or V Y, from the vanishing point, along the vanishing line, a line drawn from the point thus marked through *b* or *c*, which will give the required side on the intersecting line. (33. p. 145.)

---

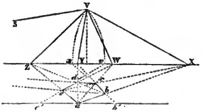
* Though the point of the vertex is supposed fixed, the vertical plane passing through it must vary as the plane of projection varies, when this latter changes its angle of inclination to the original plane.

If the given figure were to represent a parallelogram, not rectangular, segments of circles must be described on the distances between the vanishing points, which shall contain the angles, the sides and diagonals of the proposed original are to form.

If two sides of the given projection are parallel, the vanishing line must be drawn parallel to them, through the point in which the other two sides meet; and this point will be the centre of that vanishing line if the original is to be a rectangle (Def. 14.).

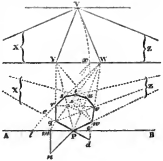
If the given projection be itself a parallelogram, the original must be parallel to the plane of it, and similar to it.

53. Let *a b* be the *projection* of a finite line, on which it is required to describe that of a regular pentagon; V X being the radial of A B, the original line. Draw the radial V Z to form an



angle of 108° with V X, then Z will be the vanishing point of the side *a e* (25): set the length of the radial X V, from X along the vanishing line to *x*, and draw *x b* produced to cut the intersecting line in *b'*; then *a b'* will be the length of the side of the polygon (33); make *a e'* equal to *a b'*; draw *e' v* to *v*, Z *v* being the length of the radial set along the vanishing line; *e v* will cut *a* Z in *e*, the extremity of the side *a e*. Draw the radials V W, V Y, to divide the angle X V Z into three equal angles of 36° each; then by drawing *a* Y, *b* Y, *e* X, *a* W and *e* W; the intersection of these lines will complete the figure *a b c d e*. If the drawing board will not admit of the vanishing point Z or X falling within it, the points *c* and *d* can be found by the vanishing points W and Y, and then by constructing the projections of the sides of the triangles *a d*, *c e*, *a c*, the point *e* can be obtained. It is very seldom that all five vanishing points will come on the board: in the present example, the side *c d* is drawn from *c* to *d* found by other means, its vanishing point being too remote, as the direction of its radial V S indicates; the radials of the perpendiculars from the angles of the polygon, bisecting each opposite side, may also be employed for finding the points, when any of the vanishing points are inaccessible. Thus if *a b* were *perspectively* bisected by a perpendicular, this would assist in determining the point *d*, and the same of any other side. The vanishing point of such a *perspective* perpendicular is found by drawing its radial perpendicular to X V, for example in the case of *a b*, the radial of the side.

Let P *q* be given, or obtained, as the projection of one side of a regular octagon, and let V X be its radial; draw V W at right angles to V X for the radial of the sides of a square on P *q*, and V Y, V Z for those of its diagonals, V Y bisecting the angle X V W, and V Z, perpendicular to V Y. Complete the projection of the square as shown in the figure, producing the diagonals indefinitely: make X *x* equal to the radial X V, draw a line from *x*, through *q*, to cut A B, the intersecting line, in *m*: then P *m* is the length of the side P Q * of the polygon (33). Make *m n* at right angles to A B, equal to P *m*, and join P *n*, which will be equal to the diagonal of the square on P Q; make P *o*, *m l*, each equal to half this diagonal P *n*, then *l x*, *o x* being drawn, will cut P *q* X, in *e* and *d*; and *e* W *d* W will be the indefinite projections of the sides of the octagon which are at right angles to P Q. The projection of the diagonals of the square on P *q*, will cut these sides in one extremity, as *s* and *v*; and Q Y, P Z being drawn will give the intermediate sides: the completion of the figure requires no explanation (See Prob. 35. p. 62, § 6.).



The *projections* of the various lines which would verify and assist the construction of the plane figure; will verify and assist the construction of its *projection*.

In these two last examples the projected sides of the polygons might of course have been obtained from the originals instead of being assumed, but the subsequent proceedings would be the same.

These examples will be sufficient to explain the mode of obtaining the projection of any plane right line figure, as the same principles are generally applicable.

---

* That is, of the original side of which P *q* is the projection.

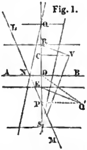
And as all plane solids consist of such figures, the projection of such a solid may be obtained by repeating the same constructions, if the intersecting and vanishing lines of the different planes of it are previously found; the manner of doing this is what must next be explained.

But the learner may be assured that unless he make himself thoroughly master of these elementary principles, he never can practically apply them; for no precept, illustrated by one example only, can be made applicable to all the combinations of lines and planes which may occur in practical perspective.

§ OF THE VANISHING LINES OF ORIGINAL PLANES.

54. Let A B be the intersecting line of any original plane, given or assumed, as before; let C be the centre of the picture, and V R the principal radial of the plane (A B) (32).

Fig. 1.



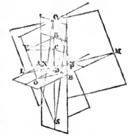
A second original plane may have three positions with regard to (A B).

I. If it be parallel to it; it can have no intersection with (A B) but its *intersecting line* * will be parallel to A B, and it will have the same vanishing line, and the dihedral angle formed by this second plane and the plane of projection will be equal to that formed by (A B) and the plane of projection, which is the plane angle C R V (32).

Consequently, to find the *intersecting line* of such a plane, it is only necessary to draw a line parallel to V R, at the same *perpendicular* distance from it, that the two planes are from each other; and through the point E, where this parallel cuts C R, a line parallel to A B will be the intersecting line required.

II. If the second plane be perpendicular to (A B); its vanishing line will pass through the vanishing point of all lines perpendicular to (A B); because all such lines will lie, either in the required plane, or in planes parallel to it and therefore equally perpendicular to (A B), and which will therefore have a common vanishing line in which that vanishing point must be situated. (18 p. 131.)

Let V R be the principal radial of the plane (A B) then V O perpendicular to V R, and consequently to the plane, will be the radial of all lines perpendicular to (A B), V O will cut the plane of projection in C R produced; because V R, V C and V O all lie in one plane which is perpendicular both to (A B) and to the plane of projection, and which cuts the latter in C R. To find S, where V O cuts C R, draw V S perpendicular to V R (in *fig.* 1.), and S where it meets C R produced, will be the



---

* The reader must recollect that the *intersecting line* of a plane, always means its intersection with the *plane of projection*, the intersection of one origin l plane with another is simply termed its intersection; he must not confound these terms.

vanishing point of all lines perpendicular to (A B): For the triangle V R S in figure 2, being turned round on its side R S; will be the triangle V R S in *fig.* 1, brought into the plane of projection[*].

Let L M (*fig.* 2.) be the intersection of the second plane with (A B); and N the point where L M cuts A B, consequently the *intersecting line* of the plane (L M) will pass through N. Let P Q (perpendicular to the plane A B) be the intersection of the plane (L M), with the plane passing through V R, V S, for these two planes being both perpendicular to the plane (A B) their intersection P Q will be also perpendicular to the same plane (Geom. IV. § 2, Pr. 18, Cor. 2.) and the point Q where P Q meets the plane of projection will also be a point in the *intersecting line* of the plane (L M), consequently N Q will be that intersecting line. To find the point Q, it is only necessary to construct the triangle D P Q supposing it turned down on the first original plane, and then brought with this into the plane of projection; in this triangle the angle P D Q is that at which the plane of projection and the plane (A B) are inclined to each other and is therefore given.

Let L M (*fig.* 1.) be the intersection given or assumed of the second original plane with (A B) and which is supposed to be perpendicular to (A B), let P be the point in which L M cuts R S, or Q D, draw P Q' perpendicular to R S, or parallel to A B; and make the angle P D Q' equal to that formed by the plane of projection and the plane (A B), make D Q equal to D Q'; then a line through N and Q will be the *intersecting line* of the plane L M: and, therefore, a line through S parallel to N Q will be the vanishing line of the Plane (L M).

If the plane (L M) were also perpendicular to the plane of projection, its intersection L M with (A B) would be parallel to R S, and consequently R S would be its vanishing line, for R S is the intersection of a plane vertical both to the plane (A B) and to the plane of projection; and consequently parallel to (L M) and is therefore the vanishing plane of (L M).

If the intersection L M of the second plane be parallel to A B, then its *intersecting line* would be parallel to A B (Geom. IV. § 1. Pr. 10, Cor. 1.) and is found by constructing the angle P D Q' as before, equal to the angle of the plane (A B) with the plane of projection, D Q being made equal to D' Q, a line through Q parallel to A B will be the intersecting line of the plane (L M) sought.

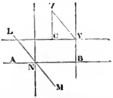
For if the two planes (A B) and (L M) be cut by a third which is also perpendicular to them and to the plane of projection, it will obviously form a triangle to which D P Q′ is equal; supposing it turned down on the side D P; and first brought into the plane (A B) and then brought with this into the plane of projection by the revolution of (A B) on A B. This will be easily understood by referring to the preceding figure.

---

[*] The learner must familiarize himself with the correct idea of the effect of turning planes round on their common intersections, so as to bring them to coincide in one plane; all the constructions of linear perspective require this proceeding. A few pieces of card board, to place in different positions, will materially assist a beginner to form correct conceptions on this subject.

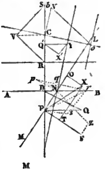
The vanishing line of (L M) is in this case a line drawn through S parallel to A B, and therefore parallel to the *intersecting line* N Q of the plane (L M).

These constructions will, in both cases, be greatly simplified, if the first original plane (A B) be perpendicular to the plane of projection instead of inclined to it, as it has hitherto been supposed to be.



Let A B be the intersecting line of this original plane, (A B,) supposed perpendicular to the plane of projection; then its *vanishing line* will pass through C the centre of the picture (21.), and will be parallel to A B; now as all lines perpendicular to the plane (A B) will be parallel to the plane of projection, these lines can have no vanishing point; or the point S will be at an infinite distance from C (or R; see last figs.); for the radial, V S, of such lines, will be also parallel to the plane of projection, and can consequently never meet it.

Let L M be the intersection of a second plane with (A B) as before, then V Y drawn through V parallel to L M, will be the radial of it, and Y will be its vanishing point; this second plane (L M) being supposed perpendicular to (A B), its *intersecting line* will be perpendicular to A B (Geom. IV. § 2. Pr. 18), and is therefore a line passing through N parallel to C V, or perpendicular to A B: and a line through Y also parallel to C V or perpendicular to A B, will be the *vanishing line* of the plane (L M). If the intersection L M be parallel to A B, the plane (L M) supposed perpendicular to (A B), will be parallel to the plane of projection; and can consequently have neither intersecting nor vanishing line (1. and 23.).



III. If the second plane (L M) be not perpendicular to the plane (A B), its *vanishing line* cannot pass through S, the vanishing point of lines perpendicular to (A B); because no line in (L M) can be perpendicular to (A B), and consequently no line in it can have S for a vanishing point.

Let L M be the intersection of the

second plane, which is supposed to be inclined to the plane (A B), at any given or known angle: from any point P[*], in L M, draw P D, P Q, perpendicular and parallel to A B, and make the angle P D Q equal to C R V, or to the angle, which the plane (A B) makes with the plane of projection; let the line D Q cut the perpendicular from P in Q. Make P O, in L M, equal to P Q; and through O and P draw lines perpendicular to L M; make the angle T P X equal to that which the second plane (L M) makes with (A B); let the line P X meet O X in X; make D Q′ equal to D Q, and draw Q′ Y parallel to A B; make Y Q′ X′ equal to the angle D P N, or to the complement of D N P, and make Q′ X′ equal to O X, draw X′ Y perpendicular to Q′ X′ cutting Q′ Y in Y, then a line drawn through N and Y will be the *intersecting line* of the plane (L M).

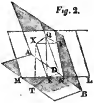


Fig. 2.

In fig. 2, these different planes and lines are shown in their relative situations, which will render the construction in fig. 1, where they are brought into one plane, sufficiently clear, the points being marked in each with the same letters.

The next step, previous to finding the vanishing line of (L M), is to ascertain the real angle this plane makes with the plane of projection. For this purpose draw P F perpendicular to L M, and make it equal to P X[†]; through F draw F Z parallel to L M and make it equal to X′ Y, join N Z; the quadrilateral P F N Z will be part of the plane (L M) turned round on its intersection L M, and the side N Z is its *intersecting line* N Y brought by this into the plane (A B); the angle Z N M being the angle made by the intersection L M, and the intersecting line N Y. (See *fig.* 2.) Make the angle Y N *p* equal to Z N M, and make N *p* equal to N P; draw *p r* perpendicular to N Y, cutting it in *q* and A B in *r*, join P *r*, make *r s*, P *s* equal to *r q*, *p q* respectively, then the angle P *s r*, or its supplement, is equal to the angle which the plane L M makes with the *plane of projection*[‡].

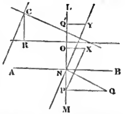
---

[*] P is taken in the line R S for the sake of simplifying the figure; but it will be seen from the construction that it might be taken anywhere in L M at pleasure.

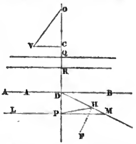
[†] That is, if the same point P is made use of, as in the last figure; but if any other point P is taken in L M, then P D must be drawn at right angles to A B, the angle P D Q being made, as before, equal to the inclination of the plane (A B) with the plane of projection; and the perpendicular must be drawn to meet the line D Q. The rest of the construction by which Y is found will be entirely changed; it will be better therefore to take P, so as to serve for both parts of the problem.

[‡] The learner is advised to draw the figure correctly to a large scale, on a piece of card board; and to cut out of another piece the triangles P D Q, P X T, and the quadrilateral P N F Z; then by setting these pieces to stand on the lines P D, P T, and P N, bringing the points Q, X and F to meet, he will perceive the principle of the

The vanishing line is drawn through *o* parallel to N Y by the construction before given (32.), making the angle C V *o* equal to the complement of P *s r* just found.



If the intersection L M were perpendicular to A B, which it would be if the plane were perpendicular to the plane of projection, then the points N and D would coincide; consequently N Q′ being made equal to N Q, Q′Y drawn parallel to A B, and equal to O X, will give the point Y at once for the *intersecting line* N Y, and a line through C parallel to N Y will be the *vanishing line* (21).



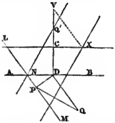
If L M were parallel to A B, let P be any point in L M, and make P D perpendicular to A B; make the angle P D H equal to the angle formed by the plane of projection, and the plane (A B), and make the angle D P H equal to that formed by the two planes (A B) and (L M); make D Q, in D P produced, equal to D H, then a line through Q, parallel to A B, will be the intersecting line of L M.

The vanishing line is found, as before, by making C V O equal to the complement of D H P, the angle formed by the plane (L M) and the plane of projection. If the plane (L M) were parallel to the plane of projection, the angles D P H, P D H would be together equal to two right angles (Geom. IV. § 2. Pr. 17. 2. in Scho.); and the lines D H, P H, would consequently be parallel, the plane (L M) having in this case neither intersecting nor vanishing line.

If the first plane (A B) be perpendicular to the plane of projection, and L M be the intersection of the second plane with (A B), from any point D, in A B, draw D P, D Q perpendicular and parallel to L M; make the angle D P Q equal to that formed by

construction in the text. He will also see that the construction for finding the angle the plane (L M) makes with the plane of projection, is founded on the principle of an auxiliary plane being supposed to pass through a line P *r*, perpendicular to both planes, cutting the plane of projection in *r q*, and the plane (L M) in *q p*; and as this auxiliary plane must be perpendicular to N Y, the common intersection of the two planes, *q r*, *q p* will be brought into one straight line by the revolution of the one plane on that intersection. This auxiliary plane will, by its intersections with the two others, measure the angle; and the triangle P *s r*, is the portion of this auxiliary plane included between the two in question, and the first plane (A B) turned round on P *r* till it coincides with (A B).

the two planes (L M) (A B). Make D Q′ perpendicular to A B, and equal to D Q, then N Q′, drawn through the intersecting point



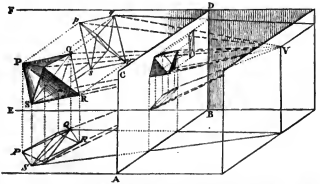
N and Q′, will be the *intersecting line* of the second plane; and a line through X the vanishing point of L M, parallel to N Q′, will be the vanishing line sought.

In all the preceding constructions, for finding the intersecting and vanishing lines of planes, inclined in all directions to the plane of projection, many abridgments of the processes will suggest themselves to the student when applying them under different conditions.

By some one of the foregoing constructions, the intersecting and vanishing line of any plane can be found, whatever may be its relative position to some other original plane, and to the plane of projection: therefore the vanishing lines of the various planes which constitute any geometrical solid, considered as an original object, may be determined; and the intersections of these vanishing lines will be the vanishing points of the intersections of the original planes respectively. For the angles which the planes of such a solid make with each other are known, or given, from the properties of the figure; and the plane of projection is taken according to the circumstances of the case, either parallel, perpendicular to, or making some determined angle with, any one or more of these planes: therefore from these data the foregoing constructions could be applied to find the vanishing lines and points of the planes and lines constituting the outline or contour of the object. But when the original solid consists of many planes at oblique angles to each other, the repeated application of these problems would cause such a confusion and multiplicity of lines on the drawing, as would render the delineation of the projection of such a figure practically useless: to obviate this, the constructions are only made use of for one or two of the principal planes, and the rest of the lines of the original are projected by the application of other methods which simplify and shorten the operation; and since the plane of projection, in by far the greater number of cases, can be taken either parallel or perpendicular to some of the principal planes, the vanishing lines of which must be determined, the simpler constructions only are generally required. Thus practical perspective becomes much easier and readier of application than the student might have been led to imagine from the preceding investigations.

A mode of obtaining the perspective projection of a *solid*, so situated with respect to some original plane, or to the plane of projection, that it would require the application of several complicated constructions in order to find its vanishing lines and points, must be explained before proceeding to any practical examples of the foregoing principles; because, though the method in question is never employed alone, for the purpose of obtaining the perspective projection of an object, yet the *principle* of it, combined with others, is applied in almost all constructions for that purpose.

Let the original solid, for example, be an octohedron P Q R S, so situated that all its faces are oblique to a given plane of projection, A B C D. The projection of the solid on any plane, by *parallel* lines perpendicular to this plane, or the *plan** of it on that *plane*, is easily determined; and if two such auxiliary planes A B E, B D F, be assumed vertical to the plane of projection, the parallel lines by which the projections *P Q R S*, *p q r s* of P Q R S, on them are determined, will necessarily be parallel to A B C D. If *P Q R S*, *p q r s* be considered as *original objects*, their *perspective* projections on A B C D, with reference to the given vertex V, being first obtained, parallel lines from the various points of these projections being drawn on the plane of projection, will obviously



---

* The explanation of what a *plan* is, will be found subsequently, under ORTHOGRAPHIC PROJECTION.

by their respective intersections determine the corresponding *perspective projections* on A B C D of the original points of the octohedron: for these lines are the perspective projections of the *parallel lines*, by means of which the plans P Q R S, $p\ q\ r\ s$ were determined; and which proceeding from the points of the octohedron, to those projections, will be represented on A B C D by the lines drawn from the points of the *perspective images* of those projections*. Since, as has been mentioned, this mode of obtaining the image of an original solid is never employed alone, from its liability to inaccuracy, it will not be necessary to give a practical example of it here, as its useful application in abridging the operations of perspective will be shewn soon, and the principle will be sufficiently obvious from the figure†, in which the *rays* determining the perspective projections of the original solid, as well as of its projections by parallels P p, Q q, &c., P P, Q Q, &c., are drawn.

---

* Before the time of Dr. Brook Taylor, this was the method chiefly employed by artists. See "Vignola's work on Perspective."

† In looking at the figure, the reader must recollect that the supposed projections on the plane A B C D are the figures the solid and its two planes would present to the eye at V; and therefore are very different to that which the reader, considered as another spectator, would see them under: the solid itself is represented by a true projection to a point, from which the reader may be supposed to view it. The same remarks apply on many occasions to the diagrams in this work.

§ ON THE PERSPECTIVE PROJECTIONS OF GEOMETRICAL SOLIDS.

For the sake of illustrating the practical application of the foregoing theorems, relating to the projections of figures and solids, consisting of right lines and planes, the mode of obtaining the perspective image of a tetrahedron will be now given at length.

The situation and distance of the vertex and plane of projection, with relation to the object, may be assumed at pleasure, being determined on, as has been before mentioned, from other considerations, foreign to the subject considered purely geometrically.

A line, A B, (Plate 1., Fig 1.) being drawn any where on the paper, for the intersecting line of the plane of the triangle D E F, one side of the tetrahedron[*], let a point, C, be assumed as the centre of the picture, from which as a centre, with a radius equal to the distance of the vertex, describe a circle, V M: let the plane (A B) be supposed inclined to the plane of projection at an angle of 70°; draw the radius C V parallel to A B, and make the angle C V R equal to 20°, (32. p. 143,) the line V R cutting C R, drawn at right angles to A B, in R; therefore a line drawn through R, parallel to A B, is the vanishing line of the plane of D E F.

Make R V′, in R C, equal to R V[†], and from V′ draw the radials of the sides of the triangle D E F, to make the same angles with each other, namely 60°, that the originals do; and to make the same angles with the parallel of the vertex that those originals make with A B (25. p. 131.). In doing this, the student must be very careful in constructing these angles on the proper side of R V′; because, from the position of that line with reference to A B, these radials will not, in this case, be parallel to the originals, as they would if R V′ had been set off, as is commonly done, on the opposite side of the vanishing line to the intersecting line; but when one of these radials, as V′ Y, is correctly determined from this consideration, the other two may be drawn one on each side of it, and making an angle of 60° with it, and all three will, if produced, cut the vanishing line in the vanishing points of the sides of D E F. (18. p. 131.)

---

[*] It must be remembered that a tetrahedron is a regular solid, bounded by four equal and equilateral triangles; and consequently if one of the triangles be given, it determines the whole figure. The same remark applies to the other regular solids.

[†] It will be readily seen, that it is immaterial on which side of the vanishing line the *vanishing plane* of any original plane is supposed to be turned down on the plane of projection: in practice, when, from the size of the construction, there is not room on the board for it to be supposed turned down on the other side from the intersecting line, as has hitherto been the case, the proceeding in the text and diagram is frequently adopted.

It will most frequently happen that one of the vanishing points (Z) will be at such a distance as not to fall within the board or paper, from the side of the given triangle, to which such vanishing point belongs, forming too acute an angle with A B (25.): in this case, the projection of that side must be either drawn through the projections *d*, *f*, of its *two* extremities, found as points in the other sides; or, if accuracy is required, one of these projections being correctly determined, the projection of the required side must be drawn through it, as tending to an inaccessible point, and the point of convergence of two given lines, the vanishing line X Y, and radial V Z (by Prob. 21. of Pract. Plane Geo. p. 41.)

But even when there is sufficient room for the vanishing point to fall within the limits of the drawing board, as the vanishing line and radial cut each other obliquely, the point is seldom well defined, and it will be better, either to apply the problem alluded to, or what is more readily done, to obtain the projections of the two extremities of the sides in question, by assuming two new original lines parallel to each other, and perpendicular to A B, consequently having the centre R for their vanishing point. The projections of these accessary lines will, in conjunction with the projections of the other sides of the triangle, determine the points in the oblique side, correctly enough to enable the student to dispense with the vanishing point.

Another construction, with which the student should be familiar, and applicable in the case in question, will be found frequently of use.

When the radial (V′ Z) is much longer than the radius of the arc[*] by which the angle R V′ Z is constructed, the intersection of V′ Z with R Z cannot be trusted to, since a very small error in measuring the arc is much magnified in drawing a long secant through its extremity. To avoid this, or to verify the construction, take R *v*, any aliquot part as half, or a third, &c., of R V′, and construct the angle at *v*, draw *v* *z*, then R *z* will be the same part of R Z that R *v* was taken of R V′, and consequently R *z* being set off the proper number of times along R Z will give the point Z more correctly, or will verify it, if previously found.

It is often advisable to apply this proceeding to a whole series of radials, drawn from the point v, and afterwards to double or treble the distances from R, of the points thus found, for the true vanishing points.

---

[*] In determining the principal vanishing points, the angles formed by the radials should always, for the sake of accuracy, be measured on the arcs of the largest circle that can be conveniently drawn, and never set off by means of a protractor; since a slight error in such vanishing points will frequently cause great confusion and trouble in the subsequent constructions.

It will, however, be frequently found, that when one side of an original polygon is very oblique to the intersecting line, or forms an angle with it of less than 10°, the plane of projection might have been so taken that that side should be parallel, and all these difficulties avoided.

The projection $d\,e\,f$, of the original side of the tetrahedron, being obtained by any of these methods, the next proceeding is to obtain that of the vertex of it. The point G, in the centre of the triangle D E F, is obviously the projection of that vertex on the plane of D E F, by a projecting line perpendicular to that plane: to find g, the perspective projection of G, or the *perspective centre* of the triangle $d\,e\,f$, draw the radials of the lines E G or F G, which radials will of course bisect the angles made by the former radials; and since the student knows this, there will be no occasion for finding the original point G at all; but he will draw these secondary radials at once, and thus find the vanishing points T, U, of any two of the lines E G, F G, D G, which may be most convenient; then by drawing e T, f U, he obtains, by their intersection, the point g as required. Draw V S perpendicular to V R, cutting R C in S, which will be the vanishing point of lines perpendicular to the plane (A B), (54 Case II. p. 159); a line, therefore, drawn through S and g, will be the perspective projection of a line from the vertex of the solid, perpendicular to the plane of the base D E F; which cuts that plane in the centre of the triangle; and the perspective image of the vertex will of course be somewhere in the line S g.

Draw G P perpendicular to E H from E; with E F for a radius, describe an arc to cut G P in P; and from H, with H E for a radius, cut the first arc in P more accurately; join H P, E P; then E P H will be the section of the tetrahedron by a plane passing through the arris from E, and through E H, and therefore perpendicular to the base; and G P will be the altitude of the solid.

A line drawn through S and T will be the vanishing line of the plane of the section E P H; for the originals of the lines g S, $e\,h$, lie in that plane, and S and T are their vanishing points; consequently S T is the vanishing line of that plane. (18. p. 131.) Find the centre O, and principal radial V″ O, of this vanishing line, (32.) by the construction shown in dotted lines in the figure; and draw T V″ the radial of E H ($e\,h$). Make the angle T V″ Q equal to the angle E H P, V″ Q cutting T S in Q; then Q is the vanishing point of the line H P, and a line drawn through h, from Q, will be the projection of this line, and will consequently cut $g\,s$ in p, the projection of the vertex of the solid.

If the angle T V″ L were made equal to H E P, then the vanishing point L,* of the line E P, ($e\,p$) of the same section of the solid would be found, and e L being drawn, it will also give the point p, and will complete the projection of the triangular section E H P, ($e\,h\,p$).

It will occasionally be necessary to apply these constructions to find the projection of a corresponding section through another angle of the original triangle; in order to verify the point p, this is done in the figure, the line S U being its vanishing line, and M the vanishing point of the arris $f\,p$.

Another mode of finding the projection p of the vertex may be employed: produce F G to its intersecting point in A B, and through this point draw the intersecting line of the plane $f\,g\,p$, parallel to S U, its vanishing line. Make S v' equal to S V′, and draw $v'\,g$ to cut the intersecting line in 1; make 1, 2, equal to G P, the height of the solid; then 2 v' being drawn, it will cut $S\,g$ in p, the perspective vertex (33.): p being thus found, lines from d, e and f to it will complete the figure, $d\,e\,f\,p$ being the projection of the tetrahedron under the given conditions.

By employing the vanishing lines of the planes of the sections E H P, &c., which, from the property of the solid, are known to be perpendicular to the plane of the original face D E F, the complicated construction of finding the vanishing lines of the other faces of the solid which are oblique to the plane (A B), as well as to the plane of projection, is avoided†. It is obvious that lines drawn through L and X, L and Y, and through K and Z will be these vanishing lines; and that L Y will pass through M, because the arris $p\,f$ is common to the two faces $e\,p\,f$, $d\,p\,f$, of which L Y, K Z are the vanishing lines (18.); and for the same reason K Z will pass through Q and M.

If the plane of projection were supposed perpendicular to the plane D E F (A B), the vanishing line of this would pass through C; let $d\,e\,f$ (Fig. 2.) be the projection of the face D E F, find the vanishing points T, U, of the lines passing through e and f, and the centre g of that face, and produce either of them to its intersecting point I; through g and I draw lines perpendicular to A B, and set off from I to K on the one the altitude G P of the tetrahedron; draw K U, or K T, cutting the perpendicular through g in p, which will be the perspective projection of the

---

* The point L is at the top of the plate; and it must be here observed, that the coincidence of V″ L and K L is accidental, arising from the symmetry of the solid, and from its assumed position.

† It will nevertheless be of use to the student to go through these constructions, by which the principles will be more firmly impressed on his memory, and generally speaking it will be advisable for him to apply all the constructions given in the course of the work, under different circumstances, and with different data, for the sake of the exercise.

vertex; for it is obvious that U I or T I is the projection of the intersection of a sectional plane, passing through the vertex of the solid, perpendicular to the plane (A B), with this plane; and since this last is perpendicular to the plane of projection, the intersecting line I K, of the sectional plane, will be perpendicular to A B, (Geom. IV. § 2, Prop. 18.) and its vanishing line will be a line through U or T, also perpendicular to A B, or parallel to I K: now U I, U K, are the projections of parallel lines, because they pass through a common vanishing point U, in the vanishing line of the plane in which the originals lie; consequently the segment $g\,p$ of a line parallel to I K, and intercepted by the *projections* of these parallels, will be the projection of a line equal to I K, or to the height of the solid, and $g\,h$ represents the perpendicular from the vertex to the base D E F.

The plane (A B) being supposed vertical to the plane of projection, the vanishing lines of the sides of the solid may be easily found, if required, by the last construction in Case III. § 35. p. 146; for the intersecting lines of the sides being found, the vanishing lines are to be drawn parallel to them through the vanishing points of the sides D E, E F, D F.

Since lines parallel to the original arrises respectively will form another tetrahedron, it follows, if lines be drawn through the six vanishing points, in the right order, these will form the projection of that other tetrahedron; for these lines passing through the same vanishing points, will represent parallel lines: this is obviously an extended application of the principle before mentioned (p. 148) relating to a plane triangle.

The student will now see how much the mode of proceeding in obtaining the projection of a given figure is governed by the particular conditions of the case; and that it would be impossible to give constant rules to be always followed—many modifications of the constructions even arising from the facilities he may possess of practically describing the necessary lines: thus if his drawing board be large, and his rules long enough, it will be better to find remote vanishing points of principal lines; and when he does not possess these advantages, he must have recourse to other methods of arriving at the solution.

In the case where the plane of projection was inclined to the plane (A B), in order to avoid the complex construction necessary for finding the vanishing lines of the sides, an auxiliary plane perpendicular to the base was made use of, the vanishing line of which was easily found. In the last example the plane (A B) being assumed vertical to the plane of projection, it would be far preferable to find the four vanishing lines as productive of more accuracy, and not requiring a complicated construction; but if the practical means did not allow of this, then another principle is to be made use of, in order to obtain the projection of the vertex of the solid. The skill and judgment of the practical draughtsman are shown in availing himself with readiness of all these various resources, according to the given conditions, and this skill can only be obtained by much thought and long practice.

The following example still further illustrates the manner in which the draughtsman may avail himself of the symmetry and properties of the solid, to obviate the necessity of finding the vanishing lines and points of planes, and lines which would require complicated constructions.

D E F G is given as the face of a cube (Pl. 2. *fig.* 1.); A B as the intersecting line of its plane; C V R the complement (32.) of the angle of (A B), and the plane of projection; therefore X Y is the vanishing line of the plane. Find the projection G *d e f* of the square, by means of its vanishing points X, Y; find S, the vanishing point of lines perpendicular to (A B); draw S Y, the vanishing line of the face of the cube; draw C V′ parallel to S Y, and C R′ perpendicular to it; draw lines from S through the four angles of the projection of the square; make S W equal to the radial of these lines; draw the intersecting line of the side of the cube through G[*], parallel to S Y (54, II. p. 159.); and by means of this, and the point W, find the projection *k* of the angle of the cube, G *k* being made *perspectively* equal to G D; complete the figure G *d e f j k i h*. Find the *perspective centres* of each of the six faces, by means of diagonals from each angle, thus *l′* the intersection of G *i*, *f k* is the perspective centre of G i, and so of the rest; draw the projections *l′ l*, *n′ n*, &c., of the perpendiculars to each face to their respective vanishing points Y, X, S.

Draw lines through *p′* to X and Y; set off half the side of the cube from G, each way along the intersecting line A B, to 1, 2, 3; set off the length of the respective radials from X and Y, to *v′ v*; then lines drawn to these points from 1, 2, and 3, as shewn in the figure, will cut G X, G Y in points, which must be transferred to *p′* X, *p′* Y, by means of the vanishing points X and Y; and from these last points, again, lines to S will cut off segments, as *l l′ n n′*, &c., from the *perspective* perpendiculars to the centres of the faces of the solid, *perspectively* equal to half the side of the cube.

In the same way *m′ m*, *p′ p*, must be also made *perspectively* equal to the same half side; either by means of the intersecting line through G, and the vanishing line of a face, S Y, or more immediately and better, by finding the intersecting and vanishing lines of the plane (G *k h e*,) passing through the diagonal of the solid; in either case, by the similar construction to that by which G *k* was make perspectively equal to the side of the cube.

---

[*] This intersecting line is only partly shown in the figure, but the construction will be easily understood.

Having found these points, and joined them and the six angles, by lines, *m h, m i, m j, m k, l* G. *l k, l i, l f*, &c. &c., drawn in the right order, the projection of the solid called the rhomboidal* dodecahedron will be obtained, produced hy placing on each face of the cube a right pyramid, the perpendicular altitude of which is half the side of the cube. It is obvious that the triangular face, as *m i k*, of one pyramid, will be in the same plane as *l i k*, that of the adjoining one; and consequently *l i m k* will form an equilateral rhomboid.

The long and complex constructions which would be necessary to find the six vanishing lines of the parallel faces of this solid are thus avoided by making use of its symmetrical properties.

And if it were required to put an octohedron in perspective, instead of finding the four vanishing lines of its faces, it will be much shorter and simpler to project the cube G *d e f h i j k*, that contains it, and then to draw the lines of the octohedron from the *perspective* centres of the six faces, as is shown in the same figure by the dotted lines *l′ m′*, *n′ o′ p′ q′*, &c.

All solids contained by three pair of equal and parallel parallelograms, such as the cube and other parallelopipeds, may be considered together, in reference to the delineation of their projection by linear perspective, since the same observations are applicable to them all. Such solids require only three vanishing lines to be determined; and if they are considered as isolated objects, as has hitherto been done with plane figures, the tetrahedron and the cube in the last example, one of these may be assumed at pleasure, and there are only two to be found by the rules already given. When the solid is a rectangular parallelopiped consisting of three pair of rectangles, the constructions required for finding these vanishing lines are the simplest the problem admits of.

But the projections of this class of solids are not only the easiest to be obtained, but are also by far the most frequently required, from the circumstance of almost all objects, such as buildings, machines, &c., the delineations of which are obtainable by geometrical rules, being composed of forms consisting of such figures: now the planes real or imaginary to which such objects may be referred, or of which they consist, are most commonly either *vertical*, that is, parallel to the direction of gravitation, or *horizontal*, that is, perpendicular to the same direction. Thus of buildings especially, the walls are upright, or perpendicular to the plane of level water; while the floors and the imaginary planes comprising the tops and bottoms of the windows, doors, &c., are parallel to the same level, the plane of projection on which these various objects are delineated is commonly assumed vertical, to avoid the strange and unnatural effect of seeing the parallel upright lines of a building projected into converging lines, which would be the case if the plane of projection were taken oblique to the originals.

---

* This solid is symmetrical, but cannot be inscribed in a sphere. It is important in the science of crystallography as frequently occurring naturally, and also as that from which the cube and octohedron can be obtained by cleavage. (See Geom. IV. Schol. p. 162.)

It might be here demanded, why it produces a more disagreeable impression to see the upright parallel lines of a rectangular object, as a building, projected into converging lines, when the *horizontal* lines are always so delineated without exciting any such feeling? This difference arises from the combined effects of our vision and our judgment: in general, the vertical lines of a building do not present themselves to our sight under circumstances which would render their *apparent* convergence so palpable as that of the horizontal lines; and as they are known to be parallel lines, we are accustomed to *imagine* that we see them as parallel; whereas we have been always accustomed to see the longer horizontal lines apparently converge towards an imaginary point, because the distance between any two such lines subtends a much greater angle where nearest the eye, than the same really equal distance does where farther from the spectator. The vertical lines are really seen under the circumstances before explained, of the top and bottom lines of a long wall, opposite to which the spectator stands; but the length of these lines, or the altitude of buildings in general, does not allow of this apparent curvature of parallel lines being so obvious; and, as has been said, we conceive that we see them parallel, because we know them to be so.

Suppose an artist, totally unacquainted with geometrical perspective, wished to make a drawing of any long building, such for example as the London University; he would place himself at such a distance as would allow of his eye seeing the whole edifice at one view, without the necessity of turning his head to look at each extremity in succession, because he would know from experience that if he approached nearer, so as not to be able to accomplish this, his view of the building would be distorted and unlike nature; and, for the sake of what is termed picturesque effect, he would not station himself immediately opposite the front, but so as to see both that and one end of the structure; to fulfil these conditions he would be at such a situation, that the length of the upright lines would not bear such a proportion to his distance from them, as to allow the apparent convergence of these parallel lines being perceivable, because his eye would be on a level with points situated between their extremities, while the horizontal lines would decidedly appear to converge, their extremities nearest his eye being possibly at not half the distance of their more distant ends. He would accordingly draw by eye, as it is termed, the outline of the building, and would make the upright lines parallel to each other, and perpendicular to the bottom of his paper, because he would know that they were upright and parallel lines: but he would draw the cornices, the top and bottom lines of the windows and the steps of the portico, *converging* or vanishing, because they would appear to do so too decidedly to admit of his being deceived as to the effect they produced on his sense of sight; and he would acquiesce in this decision of his senses against his judgment, because it was conformable to all he had ever seen of long horizontal lines.

It is commonly supposed that the whole of an object is not seen at once, if it subtend a greater angle than 60° at the eye; that is, supposing the eye to look straightforward and to remain unmoved, all rays from objects must not form a greater angle than 30° with the axis of the eye; or this being taken as the axis of an imaginary right cone, the sides of which form that angle with it, these rays must fall within the cone to admit of the object being seen, and must fall probably much within it to allow of its being seen distinctly. And an object is seen most distinctly when the axis of the eye is directed towards the centre of the mass: hence when a person looks at a building, or other large object, he stands sufficiently far from it to embrace the whole at one view, as was before mentioned; and he raises his eye so as to direct the axis of it towards the general centre. Now the most natural projection of such a building would be made on a plane of projection, assumed perpendicular to that axis of the eye; because in this case the various pyramids of rays from the component parts of the object are cut more vertically, and consequently the image most nearly resembles the apparent original form from which the rays proceed. When the building is of any altitude, this would require the plane of projection to be oblique to the vertical lines, for the axis of vision would not in this case be perpendicular to them; and these lines would therefore be projected into converging lines on such a plane, or would have a vanishing point. The image or projection thus formed would, if viewed from the true point of sight, be the best possible; but since such an image or picture is hardly ever looked at from the true vertex of the rays, but from all points and at all distances, the plane of projection must be assumed vertical to the ground, or parallel to the upright lines, to avoid the apparently unnatural representation of parallel vertical lines by converging lines, The vertical lines of buildings never can be sufficiently long in *continued* lines to allow of their convergence being seen, the laws of gravity requiring that the structure partake of a pyramidal form to ensure its stability; but if a lofty building, such as the towers of a cathedral, could be raised with long parallel lines, and an artist were to draw them, he would make them converge on his paper because he would see them appear to do so, for the same reasons that the long horizontal lines of the aisle would appear to converge; but this never being the case, all lofty buildings diminishing in width towards the top, by successive contractions at each point where a break in the vertical lines allows of it, the *visual* convergence is lost and confounded with that of the real succession of diminishing prisms; and the artist would design them accordingly, by means of short parallel lines on his drawing, for, if he did otherwise, it would be termed *bad drawing*, and reprehended as false and unnatural. The geometrical draughtsman who intended to delineate any building or similar object must arrange his point of view, or *vertex*, and the relative position of his *plane* of *projection*, so that the outline he obtains by linear perspective shall not be repugnant to the general ideas of such objects formed on the above principles.

The plane of projection being for these reasons always assumed as perpendicular to the surface of the earth, or, more correctly speaking, as perpendicular to the surface of stagnant water, the common vanishing plane of the horizontal planes of original objects, that is, of the planes *parallel* to the same surface and to one another, will be a plane passing through the eye of the spectator, and also parallel to the surface of water, and therefore perpendicular to the plane of projection. The vanishing points of all lines lying in such horizontal planes will consequently be in the vanishing line (18.) determined by this vanishing plane; hence the reason why the horizontal lines of a long building or street seem to converge towards points opposite, or level with, the eye, and why these imaginary points of convergence appear to follow his eye as the spectator raises or lowers himself above the first situation from which he looked at the object.

The surface of the earth, neglecting its irregularities, may, for the small distance at which the eye can distinguish objects, be regarded as a horizontal plane, to which the other planes of original objects are parallel: now the lines of all forms lying on the ground, such as the joints of paving stones, &c., will appear to vanish in a point somewhere in a line level with the eye, as will also all the lines lying in the other planes, such as the tops of windows, cornices, steps, &c., of buildings; but no portion of the ground, or any one plane parallel to it can be seen on the opposite sides of this imaginary vanishing line, because no rays from any lines lying on the ground or other parallel plane, can in any part of their progress to the eye, lie on the other side of the imaginary plane passing through the eye parallel to the original: * but the farther from the spectator the lines are, the nearer they will approach to the vanishing line (see p. 132); hence objects lying on the ground *appear* above one another as they are farther from the point of station of the spectator, and the farthest of those which are visible appear the nearest to the imaginary vanishing line; this is the reason why in looking at a level landscape or extensive space of flat ground, the horizon, or the last visible portion of it, appears nearly level with the eye: from the application of this principle in painting, it requires to be further elucidated.

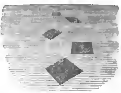
Suppose the spectator were to stand in a boat, in an open sea, in a perfect calm, so that his feet may be supposed level with the surface of the water, he would see a line all around him which bounded his view, and separated the sea from the sky, for the rays to his eye from this line would be tangential to the spherical surface of the fluid caused by the form of the earth, and consequently he could see no part of the surface beyond this line; his eye would be the vertex of a flat cone, formed by the rays from this line which would be a circle of about three miles' radius, and this small surface compared to the magnitude of the earth may be regarded as a perfect plane, the vanishing plane of which would be an imaginary one, passing through his eye parallel to it. Let any number of thin equal square boards be supposed to float in a line from him at different distances; the vanishing points of their sides would be in the imaginary vanishing line, or in the intersection of this vanishing plane with a plane of projection supposed placed vertically before him: of these boards he would see most of the surface of that nearest to him, they would diminish as they were situated farther and farther off, and one in the line of ultimate vision would appear nearly a point or short straight line; and he would see nothing at all of any supposed to lie beyond that boundary. As long as he held his head in one position he would only see an arch of about one-sixth of this circular horizon at once, and from the small proportion which the height of his eye above the level of the sea, or original plane, bore to the distance of that horizon, it would *nearly* seem to coincide with the imaginary vanishing line of the boards, as is exemplified in the annexed figure, the curved horizon being produced, not as is commonly though erroneously supposed, by the sphericity of the

---

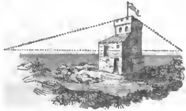
* In speaking of really visible objects, the hypothetical case of rays from objects behind the eye, or vertex, which determine projections on the other side of the vanishing line, is of course out of the question.

earth, but by its being a circle which is looked at from a point not in its plane.

Now if the spectator in such a situation could elevate himself to a greater height above the surface of the sea, every thing else remaining the same, his circle of limit of view would be extended, because the rays forming the side of the cone would touch the surface at a greater distance: the farthest board, which from below was only just visible, and which appeared touching the horizon, would now appear within, or below, it; or he would see other boards floating beyond it, which were before hid from him, and which would now appear as it were *above* the former, or nearer to the imaginary vanishing line in its new position: for the vanishing plane still passing through the eye would rise with the spectator; the surface of the intermediate boards would be less *foreshortened*, as it is termed by artists; because the rays proceeding from them would form greater angles with the plane of their surface or of the sea: but if the increased altitudes of the spectator's eye were considerable, the visible horizon would differ very perceptibly from the real one or the vanishing line; though still, the difference between them being but small, compared to the much more increased visible surface of the sea, they might be imagined to coincide; and in the case supposed, the spectator having no near object, with which to compare them, they would probably appear to him identical.



But suppose him to stand on the edge of a lofty cliff on which a building was situated; he would, by accurately considering the horizontal parallel lines of this, perceive that their vanishing point was in an imaginary line level with his eye and *above* the visible horizon of the sea, as will be understood from the annexed figure, where the real horizon is shown by a dotted line, with the visible horizon below it*.



---

* This difference between the real and imaginary horizon is termed, astronomically, the *dip of the horizon*, and the angle of the cone of rays from the circle bounding the surface in these examples, is termed the *angle of depression*.

The same apparent *rising up* of the visible horizon, and the foregoing observations on it, of course equally apply to the dry ground; but from the natural inequalities of this, and from the circumstance of distant hills, &c., preventing the real visible horizon from being generally seen, the subject was explained by the hypothetical case, for the purpose of rendering it more clear.

Artists conversant with the subject, in pursuance of these principles, assume in their pictures a horizontal line for the vanishing line of all original lines of buildings, &c., parallel to the ground, and in drawing their *distance*, take care, if a flat level country is to be expressed, that the visible horizon be shown rather under this line; in general, however, in pictures, the distance is formed by hills, in order to avoid the unpleasing uniformity of a straight line; as these hills are, correctly, delineated as arising above the natural horizon, or even the vanishing line of horizontal planes, this distinction is not observable; but many of the finest painters, being ignorant of the geometrical principles of perspective, the most flagrant violations of correct drawing in this respect are constantly to be seen in the best works of art[*].

The absolute size, or area, of the *projection* of any solid, or plane figure, depends on the distance from the vertex at which the section of the pyramids of rays is made by the plane of projection, that is, on the *distance* of the vertex (Def. 7.): though at whatever distance the section is made, provided the plane is supposed parallel to its former positions, the figures of the projections will be *similar* (Geom. IV. § 3, Schol. to Prop. 29). Hence an image, or projection, of any size, may be conceived to be produced from an original object, such as a building of whatever magnitude this may be; but such projections could not obviously be obtained by the rules already given from the building itself, since the dimensions of this preclude the possibility of their being made use of on the drawing board; the method of practically getting over this difficulty must now be explained.

If all the rays proceeding from an original object to the eye, or vertex, were divided by points in any, the same, ratio, and these points were supposed joined by lines in the same order as the lines of the original, from which the rays respectively proceeded, an imaginary solid would be formed precisely similar to that

[*] Claude Lorraine, who understood *aërial* perspective perhaps better than any other painter, perpetually drew his buildings in the most preposterous *linear* perspective; as may be seen from the pictures in the National Gallery. Such inaccuracy is not so pardonable in pictures where it is not redeemed by the intellectual and practical beauties of this and other great masters: any further consideration of this would however be misplaced in a work on practical linear perspective.

original, the different planes and lines of which would be parallel to those of the original ; and this solid might be substituted for the building, or other object itself, in order to obtain the projection or image of the latter, since the rays from each are identical, and consequently would produce the same figure, by their intersection with a plane of projection. This is what is in effect done by the artist, when, in order to obtain the outline of a building, or other original object, by linear perspective, he makes use of the *geometrical plans* and *sections* of it, made to any convenient scale, according to the size he intends his drawing to be ; for these plans are nothing else than the projection by parallel lines, of such an imaginary solid or *model* of the original, made upon planes parallel to those of the solid itself: thus a *ground plan*, as it is termed, of the London University, may be made to such a scale as to be comprised in the page of this book, and this plan may be conceived as the projection on a plane parallel to the ground, of a model precisely similar in every respect to the edifice itself.

The mode of putting the outlines of buildings in perspective, according to the foregoing principles, will now be shown.

In the first example (Pl. II. *fig.* 2.), the general plan of the house, D E G . . . L, is supposed to be drawn on the paper on which the projection is to be obtained ; this is done when there is sufficient space on the drawing board, and when the outline is not too complicated ; for otherwise, as this plan is to be subsequently rubbed out when the projection is completed, the numerous lines of construction would injure and soil the drawing.

This outline plan must be drawn so that the image may be upright or *square* on the drawing ; accordingly the intersecting line, A B, is always first determined on, and then the plan annexed to it, as in the figure, so as to fulfil the necessary conditions.

The intersecting line A B of the plane, on which the imaginary model of the building is supposed to stand, being drawn to form the proper angles with the principal lines of the plan of it, D I K L, and touching one angle D if convenient, draw the vanishing line, parallel to A B, at the distance of about six feet to the scale of the plan, and mark a point C in it for the centre of the picture.

The point C, and the line A B, must be taken so as to give the required view of the building according to the observations in the next example (p. 185.).

Draw C V, perpendicular to A B and the vanishing line, and equal to the distance of the vertex : draw the radials of the principal lines of the plan, to find the vanishing points X, and

Y * of the horizontal lines of the building; draw D M perpendicular to A B, and through any point M in it, taken at pleasure, but as far from A B as may be convenient, draw a line N O parallel to A B, for the intersecting line of an auxiliary plane parallel to (A B), on which the plan of the building is to be supposed drawn, and which plan will therefore be seen in perspective: find this projection M *i k l* of the given plan D I K L by means of the vanishing points and rays, by the methods already explained for the perspective projections of rectangles. Through the points of this perspective plan, draw lines perpendicular to A B as required, to determine the corresponding points in the projection of the general form of the building. Make D T, in D M, equal to the height of the walls from the ground to the under-edge of the thatched-roof, as given by the elevation †. From D and T draw lines to the vanishing points X and Y to meet the perpendiculars from the perspective plan in *i, i, l, l,* &c.: in this way the projection of the rectangular solids representing the body of the building is obtained; as will be easily understood from the figure by those who have made themselves masters of the preceding precepts.

Draw lines perpendicular to the vanishing line X Y, through those vanishing points; these lines will be the vanishing lines of the upright faces of the building, and the next step is to find in them the vanishing points of the oblique lines of the roof, such as those which form the gables. Set off the lengths of the radials V X, V Y, from X and Y along the vanishing line of (A B) to W and Z; at which points draw radials to form the same angles with the line X Y, that the planes of the roof form with the horizontal planes; these radials will cut the perpendiculars through X and Y, in the vanishing points of the sloping lines of gables sought.

For, supposing the vertex in its original situation with respect to the plane of projection, it is obvious that the radials of the oblique lines of the gables will form the same angles with those of the horizontal lines of the walls, that the original lines themselves do; and then supposing the vanishing planes of the walls brought into the plane of projection by revolving on the vanishing lines, the vertex will be brought into the plane of projection to the points W and Z, and the angles alluded to will be represented there of their correct magnitude, and are therefore constructed so accordingly. Produce Q R, S T, the plan of the roof ridge, to the intersecting points, a and b, make $a p$, $b p$, perpendicular to A B, equal to the height of the ridge from the ground, by the elevation, or scale. Then lines drawn from p, p, to X and Y will give the ridge in the projection, and the points $q\ r\ s\ t$, &c., in them, are to be determined from the perspective plan above.

---

* The vanishing points do not fall within the plate; but their situation is immediately seen from the various dotted lines.

† The elevation is not shown, nor is it necessary that it should be drawn, provided the artist knows the dimensions of it, so as to take these from the scale at once when required.

Draw, in the perspective plan, the lines indicating the projection of the eaves corresponding to the lines D′ I′, D′ L′, &c., in the original plan. Draw $f x$, $g x$ (see *fig.* 3.) to their proper vanishing points in the vanishing line passing through X; $f x$, $g x$ will intersect in the *perspective central* line $x h$ of the wall $f f$, $g g$: this central line is found by the perspective diagonals $g f$, $g f$; draw $x x'$, $f f'$, $g g'$ to Y. Find the vanishing points, in the vanishing line through X, of the lines which are at right angles to the originals of $f x$, $g x$; draw $f o$, $g o$, to these points, and mark the points o by means of perpendiculars from the corresponding points in the perspective plane above; draw $o t$, $o t$, to the respective vanishing points; the quadrilateral figures $f o x t$, $g o x t$, will represent the thatch cut through by the plane of the wall; and by means of them, the front edge $f' o' x' t'$; $g' o' x' t'$ of the thatch as really seen projecting beyond the face of the wall is easily drawn, the projection $t t'$ being also obtained from the perspective plan above.

From this, the mode of proceeding in finding the perspective projection of the eaves in the other parts of the view will be easily comprehended; but it would be useless to enter into further details, since so much depends on the peculiar circumstances of the case, that it is impossible to give any precise rules. If the scale of the drawing be very small, it would only be necessary to find the principal points, and the projection of the eaves might be drawn in by eye; which, if the draughtsman be accustomed to drawing from nature, he will be able to do more correctly than he would by applying the geometrical rules; but in old buildings, or in modern *picturesque* cottages, the eaves project much and are often ornamented, so that it was necessary to explain the mode of obtaining the correct outline of them.

The plan of the *dormer* window in the roof must also be drawn in the perspective plan; and having marked from this the point v in the principal ridge of the roof, where that of the *dormer* meets it, draw $v y$, to the vanishing point in the vanishing line through Y (see *fig.* 4.), and complete the quadrilateral representing the section of the thatch by a plane $v y y$ parallel to the end wall of the house, D L, and passing through the ridge of the roof of the *dormer*; the point z in the lower line, where the bottom of the upright front of the window cuts this quadrilateral section, is deduced, as before, from the plan above: through z draw $i i$ to X, and draw the upright edges of the front $i k$, $i k$, from the plan; the height of the line $k k$ must be set off along D M from the elevation, and transferred by X to the line *y p*, in which the plane *w v y* cuts the front wall of the house; then it must be transferred from *p* to *q* in the intersection of the same plane with the front of the window; and *k k* must be drawn to X through *q*. The mode of finding the lines of the roof and projecting eaves of the dormer, is precisely the same as that described for the gable, and does not require repetition.

The heights of the tops and bottoms of the windows, doors, &c., must be set off along D M from the scale, or from the elevation, and transferred by means of the vanishing points along the faces of the walls, as is shown by the dotted lines. And the beginner must be cautioned against laying off any dimension from the scale, or from the elevation, on any line of the perspective projection, unless it coincides with the plane of projection: this is the case with the corner of the building D T, and hence all dimensions must be set off on that line, or on the two intersecting lines *a p*, *b p*, because the size of the original objects can be shown on those lines of it alone which coincide with their projections (46 p. 151). Thus the height of the windows in the gable from the ground must be either set off on *b p* and transferred by Y to the central line of the wall, or must be set off on D T to 1, 2, 3, 4, &c., and first transferred by X to *e e*; then by Y to *f f*, and then by X, again, to the same central line. Again, the height of the top of the chimney from the ground must be taken from the scale and set off on *a p*, and then transferred by X to the central line of the front of it, as obtained from the plan, because as will be seen from the plan and elevation, the planes *a p*, *b p* are those in which the central lines of the end of the building and the chimney and the central line of the gable are respectively situated.

To avoid the multiplicity of lines in the perspective plan, the plan of the bow window is drawn on another horizontal plane, the intersecting line of which is assumed, parallel to A B at pleasure through any point M′ in D M. The oblique sides of the bow are found by the original plan to form an angle of 135° with the front wall, consequently their vanishing points are obtained by drawing the radials V X′, V Y′* to bisect the angles formed by V X, V Y. The upright lines of the bow being got from the plan of M′, the horizontal lines of it are to be drawn to their respective vanishing points Y′, X and X′, as shown in the figure.

In this example the principle before explained (p. 165) is applied; by drawing the geometrical plan, considered as an original object, in perspective; the auxiliary plane being taken perpendicular to the plane of projection, or parallel to that on which the building (or the imaginary model) stands. Hence the parallel projecting lines, by which the geometrical plan is obtained from the building (or model), will be projected into parallel lines, because the originals are parallel to the plane of projection; and therefore, if the geometrical plan be put in perspective, as it would appear under the given conditions of the situation and distance of the vertex, the *picture* of the building or model, as this would appear from the same point of view, can be obtained by means of it, as is here done.

---

* Y′ does not come within the plate.

It is obvious that the perspective plan might have been drawn on the plane A B; but the eye being but little elevated above that plane, it would be so much foreshortened, that no points could be accurately determined from it, on account of the oblique intersections of the lines, as will be immediately seen by comparing the very oblique angle *f e* D with the same angle at *f e* M, both of these being projections of the right angle F E D. Added to which the number of lines required in the perspective plan, would, if it were drawn at D *e f g i l*, &c., confuse the figure and injure the paper; whereas if this auxiliary figure be carefully drawn lightly in pencil at M, or any where above or below the figure, it can be afterwards rubbed out without injury to the drawing.

The size of the figure in the plate does not admit of the subordinate constructions being shown, for drawing in the window cills, sashes, architraves, &c. &c., and it must be remembered that, except in very peculiar cases, these minor parts of a design are always best drawn by eye, which, if the artist be a tolerable draughtsman, will be eventually a more accurate as well as a more expeditious method of proceeding, than that of continuing the application of geometrical principles to such minutiæ. In proportion as the artist is skilful in the use of his pencil will he content himself with finding the more general and important lines of his design, and will trust to his judgment for filling up the details; and even these more important ones should not be put in, on a drawing intended to become a picture, with hard stiff lines, but should be gone over with the pencil, to impart to them that picturesque inequality of strength and direction, which those even of the newest buildings have in nature. For it must be borne in mind, linear perspective is only an application of geometry to the *fine arts*, and is intended merely to obtain a more correct outline of certain objects than could be done by the judgment and eye unaided by *rules*; the outline so obtained is to be subsequently treated just as that of any natural object drawn without such assistance.

But when the building is complicated and large, the mode of proceeding in the last example is objectionable; not only because, with all possible care, the drawing in such a case would be injured by the numerous accessary lines required, but also because many lines cannot be determined sufficiently accurately, and may be more expeditiously determined by the following method

Draw a general plan of the building as D E K . . . . N[*], &c., (Pl. III. *fig.* 1.) marking on it the projection of the cornices, the hips and ridges of the roof, the chimneys, windows, doors, and every principal external part that can be seen in the proposed view, as is shown in the figure. In practice this is best done on a separate drawing board, so that the paper on which the finished outline is to be made may not be injured.

Having determined on the point V (towards the bottom of the plate) from which the best or most convenient view can be obtained, according to the form and size of the building, and so that the whole of it can be seen under an angle of not more than 30° to 40°, draw a line A B, to represent the plane of projection, so that C, the centre of the picture, shall not be very far from the general centre of the mass of the object; for, if this condition be not attended to, the representation of the building will be distorted, owing to the pyramids of rays from some parts of it being cut much more obliquely than others.

The *direction* of the plane of projection, with reference to the plan, being determined from these considerations, its *situation* with respect to the eye, or object, may be assumed with the sole view of facility of construction in the subsequent operations; for the form of the projection, or view, of the object depends, as has been shown, on the situation of the vertex, and by moving the plane of projection nearer or farther from that point, parallel to the same direction, the magnitude of the figure only is changed. It is therefore advisable to take A B, so that one of the principal angles, or vertical lines, of the building shall be in the plane of projection, which allows of the altitudes of the different parts of it being set off at once on that line, as was done on D T in the preceding example. In the present case, the artist would therefore draw A B touching the angle D nearest to V: draw from V, the radials V Y, V Z, parallel to the principal lines of the plan, in order to determine the distances of Y and Z, the vanishing points of the horizontal lines of the building[†], from the centre C.

Now the projecting planes (Def. 5.) of all the lines of the object which are perpendicular to the plane on which it stands, and therefore parallel to the plane of projection, will pass through the point V; V being supposed to be the point where a perpendicular from the vertex meets the original plane (A B), or the point on which the spectator stands: the intersections therefore of these projecting planes with that original plane (A B), will be lines drawn from the various points of the plan to V; because the plan being a projection on that same plane by lines vertical to it, these points will represent the vertical lines of the object. The intersections of the same *projecting planes* with the *plane of projection*, will be the lines of the perspective image of the object; consequently, if the different points where the lines to V cut A B in *fig.* 1., be transferred by means of a slip of paper (see note, p. 32) to their correct place with respect to the centre C, to the intersecting or vanishing lines of the original plane in *fig.* 2., the indefinite projections of the *upright* lines of the building, and of all other lines of construction by which the plan was determined, will be obtained at once by drawing lines perpendicular to A B or Y Z through the points thus set off.

---

[*] The whole plan is not shown in the plate, for want of room, nor need more of it be drawn in reality, than is absolutely requisite.

[†] The points Y and Z, where the radials meet A B in fig. 1, and on the horizontal line in fig. 2, do not fall within the plate; but the text will make the figures clear.

The next step is to prepare the drawing board considered as representing the *plane of projection*; to do this, draw two parallel lines, *fig.* 2., at the distance from each other that the *vertex* or point of sight is supposed to be elevated above the original plane A B on which the object stands, and on which the plan, *fig.* 1., was made; the bottom line A B will be the intersecting, and the other Y Z the vanishing line of that plane. Take C, the centre of the picture in the vanishing line Y Z at pleasure *, and set off from it the vanishing points Y, Z, at the distances C Y, C Z, from *fig.* 1. By means of a strip of paper, take off from A B, *fig.* 1., the points in which the lines, drawn to V, cut A B, and set them off along Y Z in their correct situations from C as given on the plan: and through the points thus marked draw by the T square, the upright lines of the building, perpendicular to Y Z. The heights of the upper and lower lines of the windows, doors, cornices, &c., are to be set off on D L, and transferred by means of the vanishing points to the various fronts of the building; but as this causes a good deal of trouble when it has to be done for the purpose of obtaining a line in one part only, as, for example, the top of the pediment at *p*, it is advisable to obtain the intersecting lines of imaginary planes, passing through the central lines of the main body of the edifice, or of the wings, and then the dimensions taken from the scale can be set along these and transferred at once to the part where they are required: this proceeding is shown in the figure with regard to the ridge of the roof, the pediment, and the chimneys; A R, S T, U W being the intersecting lines of such planes, as will be understood from the plan.

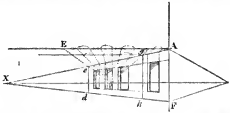
The upright sides of the windows, &c., are obtained from the plan in the same way as the other upright lines *, but it is better to get the principal lines of the building in, before the subordinate parts are attended to.

---

* When practicable, it ought to be taken so that the two principal vanishing points will come on the drawing board.

Another mode of finding the projections of the upright sides of a set of windows, or other upright lines, which is frequently used, must be here noticed.

Let A F *e d* be the side of a building, or other object, which is to be divided *perspectively* into any proportional parts; as for windows, &c., and the intervening piers; or for any similar purpose.



From A, the extremity of the line A *e* to be divided, draw a line A E parallel to the vanishing line of the plane in which A *e* lies, and from any point, P, taken at pleasure in the *vanishing line*, draw P E through the other extremity *e* of the perspective line, cutting A E in E; divide A E *geometrically* into the segments, in the proposed ratio; then lines drawn to P from the points of division will divide A E *perspectively* into the same segments, that is, the segments of A *e* will be the *projections* of the geometrical divisions on A E.

For A E being parallel to the vanishing line, is either the intersecting line itself, or a line parallel to the intersecting line of the plane in which the original of A *e* lies; consequently, the original of A E is parallel to the plane of projection (26. p. 132). Now P being taken in the vanishing line of the plane of A E, and A *e*, the lines drawn to P, represent parallel lines, the originals of which would divide the originals of A E, A *e*, in the same ratio (Geom. I. § 7, Pr. 54.); therefore the segments into which the *projections* of these parallels divide A *e*, will be the *projections* of segments in the same ratio as those of A E, and this last line being parallel to the plane of projection, would be divided in the same geometrical ratio as its original (44. p. 151.).

---

* The draughtsman, in practice, will save himself much trouble if he stick pins, or, still better, fine needles, into the points V, and the vanishing points, to which a great many lines are required to be drawn; he will then, by keeping the edge of his rule against the needles, be able to draw a great many lines in succession without having to adjust it to two points, as he otherwise would have to do: these needles must be stuck very upright, or the rule will not apply to the true point. The line A B on the plan had better be drawn in with ink; then the points being transferred to it with a pencil, they may be effaced when so numerous as to cause confusion; this, of course, can only be done on the supposition of the plan being on a separate board.

If the point A be the *intersecting point* of A *e*, then A E is the *intersecting line* of the plane (6. p. 126.); but if *g e* were the perspective segment to be divided, a line drawn from *g* parallel to the vanishing line is to be employed; the divisions of such a line being still in the same geometrical ratio as the corresponding segment of the intersecting line cut off by the perspective parallels, drawn from P through its extremities.

If the distance X P from the vanishing point to the assumed point were equal to the *radial* of A *e*, then A E would be equal to the original of A *e* (33.) and the divisions of it might be set off on A E from the scale; but as P is taken at pleasure, and, in general, so that the lines drawn to it may not cut the perspective line to be divided too obliquely for accurate division, A E may be either less or greater than the original of A *e*; but still the divisions of the former will be in the same absolute ratio to the whole line and to one another that the divisions on that original would be.

This method is applicable when the separate plan (*fig.* 1.) of the present example has had so many lines and points marked on it, as to create confusion by attempting to deduce the upright lines of the windows from it; but it is still more useful when the general outline of the mass of the object has been obtained without a plan at all, but by means of the dimensions of it set off from the intersecting points along the intersecting lines, and transferred to the projections of the originals by the application of the principle explained p. 145.

It will not be necessary to find the vanishing points of the oblique lines of the roof, because the points where these cut the *ridges*, or horizontal lines, are better and more quickly obtained from the plan, and the projections of the oblique lines can be drawn through them; but it will be advisable, if space admits of it, to find the vanishing points of the sloping sides of the cornice of the pediment, in order that the lines of the mouldings may be drawn correctly converging: these vanishing points are of course in the vanishing line of the upright front, drawn through Y, perpendicular to Y Z; and the points are to be obtained by constructing the angle of inclination of the pediment, at the extremity of the radial V Y, brought into the plane of projection on the vanishing line Y Z, as was explained of the slopes of the gables in the last example.

In these two examples, the principal vanishing line, or that of the horizontal original planes, would represent the limits of the visible ground, or plane, on which the buildings stand, or would represent the *horizon:* accordingly this line is commonly called the *horizontal line*, and is level with the eye of the spectator supposed to look at the object; as the spectator commonly stands on the ground, the height of the horizontal line above the intersecting line should be about six feet on the scale to which the plan of the building is drawn—that being the ordinary height of the eye, or rather above it; but it should not be less than that, in any case, or else the bottom lines of the image of the building would be too nearly in a continued line, and the pavement of the street, or other objects on the ground, would be too much foreshortened. If the horizontal line were taken exceeding six feet, it would give an erroneous idea of the size of the object; it would appear of a smaller size, because, being accustomed to stand on the ground to look at buildings, &c., in nature, we therefore see the horizon, or level of the eye, at a tolerably constant altitude, and consequently make use of it as a species of scale to estimate the magnitude of an object represented on a drawing. If the horizontal line were taken greatly exceeding six feet, the drawing would then directly indicate that the spectator had been much elevated above the ground. As artists constantly fall into the most glaring errors on this subject, particularly when they introduce figures in their drawings, this requires to be more fully explained.

The human figure being generally of the same average height, if a person stand on the level ground, as in a street, he will see the heads of other persons in it on a level with his own, and if he were to draw the street and the figures he saw, his sketch would present the appearance shown in *fig.* 1., where the heads of the different figures are all on a level, but the figures are smaller in proportion as they are more distant from the spectator, the nearest one standing on the line on which the plane of projection is assumed to stand. If the artist meant to represent a person still nearer, or between him and the plane of projection, he would show his head level with those of the other figures, and his feet would come below the intersecting line; but all these figures would represent men of the

Fig. 1.

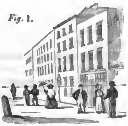
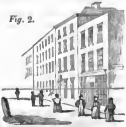


Fig. 2.



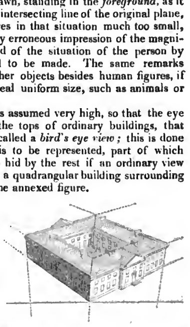
ordinary height standing on the same level plane as the spectator; and would therefore afford a sort of scale by which the height or magnitude of the buildings and other objects might be estimated.

Now suppose the buildings, the height of the vanishing line, and the place of the vertex or point of sight to be the same, and therefore presenting the same projections as before, and the artist were to draw the figures as shown in No. 2. Any other person, looking at such a drawing, would refer to the figures for a scale, and perceiving that the visible horizon was above their heads, he would conclude that the artist stood elevated above the level of the heads of the figures; and therefore he would suppose the buildings to be much larger than they really were.

This is the error commonly committed by artists, who, in order to avoid the sacrifice of the great space a figure would occupy, if correctly drawn, standing in the *foreground*, as it is termed, that is, near to the intersecting line of the original plane, or ground, design their figures in that situation much too small, and by this means give a very erroneous impression of the magnitude of the other objects, and of the situation of the person by whom the view is supposed to be made. The same remarks would of course apply to other objects besides human figures, if these are generally of one real uniform size, such as animals or trees.

When the horizontal line is assumed very high, so that the eye of the spectator is above the tops of ordinary buildings, that species of view is produced called a *bird's eye view;* this is done when a mass of buildings is to be represented, part of which would be in a great measure hid by the rest if an ordinary view were taken, as, for example, a quadrangular building surrounding a court-yard, as is seen by the annexed figure.

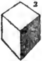
But in such a view the plane of projection should not be assumed perpendicular to the ground, because if the upright lines of the building are represented by parallel lines, the image looks distorted or unnatural; since a person looking at a building from such a point of view would observe the convergence of the upright lines too distinctly, not to have his eye hurt by seeing them represented in a picture as parallel; this must be especially attended to if the buildings are lofty; and the plane of projection, to produce a natural and faithful representation, must



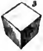
be taken as vertical to the ray from the general centre of the mass; by so doing the upright lines of the object will have a vanishing point below the intersecting line of the plane on which it stands.

1

2



3



This principle is exemplified in the annexed figures, representing the same cube, seen from two different points. In No. 1., the height of the eye above the plane on which the cube stands is not much greater than the height of the cube itself; the plane of projection is therefore taken perpendicular to that plane, and the image of the object is *natural*, or nearly as it would really appear. In No. 2., the eye, or vertex, is very much elevated above the original plane, and the plane of projection being still assumed perpendicular to this, the upright lines are still parallel: the consequence is that the image is distorted, and does not convey the idea of a true cube, although it is a perfectly correct projection of such a solid under those conditions. In No. 3., the place of the vertex is precisely the same with reference to the cube as in No. 2., but the plane of projection is perpendicular to the ray from the centre of the object, and consequently oblique to the upright lines of it: these have therefore in this case a vanishing point, and the image is *natural*, because the real cube would present such an appearance to a spectator, who would naturally direct his eye towards it; whereas in the former case the eye is supposed to be directed towards the centre of the picture, and the cube looked at obliquely.

These remarks may be verified by any person who looks attentively at the effect produced to his eye on the adjacent houses, when viewed from the top of the Monument, or from St. Paul's; and he will find that if he attempt to draw these houses from such situations, by eye, he must make the upright lines of them converge, to represent them correctly or as he really sees them.

The method of drawing a "bird's eye" view of a building is the same as that for an ordinary view of any object. The vanishing point of lines perpendicular to the original plane being found (54, p. 159), the upright lines of the building, and all lines of construction, representing lines parallel to them, must always be drawn to it instead of parallel. The only particular observation that is required here on the subject, is, that if the second mode of proceeding by a separate plan be adopted, as in the last example, (p. 185) the points in which the lines, drawn from the points of the plan to V, intersect A B, must always be set off on the intersecting line, and not on the vanishing line of the plane on which the plan is drawn, nor on any other parallel to them, as is done when the plane of projection is perpendicular to that original plane; because in the latter case the vertical lines of the object being projected into *parallel lines*, these are everywhere at the same distance apart, and these distances, as deduced from the separate plan, may be laid off on any line parallel to A B; whereas, on the contrary, in the supposition of the plane of projection being oblique to the original plane (A B), the lines of the object vertical to this plane are projected into converging lines, and it is only on the common intersection of the plane of projection, and the original plane as represented by A B, (see figs. in Pl. III.) on the separate plan, and on the figure of the projection, that the intersections of the *projecting planes* with the original plane, and with the plane of projection, are at the same distance apart. Strict attention must be paid to this, or the artist will be led to commit great errors in the commencement of his drawing, which will cause him much trouble to rectify afterwards.

The constant occurrence, in architectural subjects, of cornices, or collections of mouldings, projecting from the faces of the building, renders it necessary to give an example of the mode of finding their projections in perspective drawings.

Let D L, E M, F N (Pl. III. Fig. 3.) represent the angles of a building, in a projection; the line K L M N . . . being the horizontal bottom line of a cornice, of which figure 4 is the geometrical elevation and profile. Make L P equal to *l p* the depth of the cornice, supposing the arris D P to be in the plane of projection, draw O P Q R . . . . the line of the top of the cornice, to the respective vanishing points. Make P S, P S′, in Q P, O P, produced, *perspectively** equal to the projection of the cornice *p s*; set the depths of the various mouldings along L P, and determine the lengths of their projections on the lines drawn to the vanishing points through the points in D P: this operation will be much facilitated by drawing *l s*, L S, L S′, which will in many instances determine the points of the mouldings at once; as, for instance, the top and bottom of the fillets of the ovolo, and unless the scale of the drawing is very large, by means of *l s*, and its perspective images L S, L S′, &c., all the mouldings can be proportioned in the figure by eye. In this manner the *perspective projections* L P S, L P S′ of the sections of the cornice by the planes of the building D M, D K, supposed to be extended for that purpose, may be obtained; and it is clear that lines drawn to the proper vanishing points, through the points of these sections, will give the image of the cornice mouldings as they would appear standing out from the faces of the walls.

---

* Here again this term is used as signifying that P S, P S′, are to be made the perspective projections of *p s*.

These lines by their intersections will give the *mitre* L P T; but if the scale is large, it will be necessary, or advisable, to find these mitre sections at each of the principal angles of the building, as shown in the figure by L P T, M Q W, &c. The planes of these mitres form angles of 45° with the sides of the building, consequently the vanishing points of the lines P T, Q W, &c., are found by radials bisecting the two right angles formed by those of the lines P O, P Q, &c.*

It is obvious that these mitre sections of the cornice, if found at first, would supersede the necessity of drawing the two square sections L P S, L P S′ at all; because lines drawn from the mouldings, as found on the mitre sections, to the vanishing points, will give the perspective projection of the cornice at once; and in practice this is usually the mode of proceeding, for a tolerable artist can, in general, proportion by his eye the mouldings, as seen in perspective, accurately enough in most cases, if the scale is not very large; but if it be, or precision be required, then the mode of proceeding by the square sections is advisable, because, from the situation whence the cornice is viewed, the mitre sections are too much foreshortened to admit of the mouldings being determined on them with accuracy. If the mitre sections alone are employed, it is obvious that the geometrical profile *fig.* 4 will not be the same as that formed by the oblique section of the cornice; and therefore this last must be obtained from the plan and elevation of the mouldings, in the manner shown in *fig.* 5.

In order to avoid the necessity of finding the square section made by the plane E M F N, at the angle N F, it is drawn on the plane S′ P L, it being more easily determined there, by prolonging the lines by which the section S′ P L itself was found: thus the lines S′ S″, L N″ are to be made *perspectively* equal to the depth of the break in the building N M; and by means of the line S″ N″ the mouldings of the cornice on the face of the wall F G, as supposed prolonged to S″ N″, can be drawn; and therefore, conversely, the perspective cornice can be drawn from this imaginary section if it be previously found.

The vanishing point of the raking mouldings, which form the pediment, must next be found; and the perspective section of these by a vertical plane taken any where, where most convenient, must be obtained. This plane is generally assumed as passing through the apex of the pediment, and consequently forming the mitre of the angle of junction of the two sides; but as this could not be done in the present example, the plane is shown as passing through the line $n\,n$, (*fig.* 4) which passes through the lower angle of the *tympanum*, or the plain part of the pediment.

---

* This will be immediately understood from *figs.* 1 and 2. The radial of the mitre plane will bisect the angles of the radials V Y, V Z.

The artist must be reminded, that from the inclination of the mouldings of the pediment, as they are of the same depth and project as much as the horizontal parts, they will not coincide with the mitre section of the horizontal cornice at N R, as might at first be imagined; and consequently that mitre section, if found at N R perspectively, cannot be made use of for drawing the perspective projection of the pediment cornice, except in the case of the upper line of the bead or fillet above the *corona*, which from the architectural construction of the pediment will coincide at this mitre, as is seen from *fig.* 4; from which also it will be perceived that the point w does not coincide with s, and therefore W′ w in the perspective image cannot be drawn through W, the point corresponding to s, in the mitre section M Q W.

N N′, in the line N G, is to be made perspectively equal to $l\,n$ (*fig.* 4.): then the whole depth $n\,n$, and that of the different mouldings on the oblique section, being set along D P produced, they must be transferred to N N′ by means of the vanishing points, as shown in the figure. The distance N′ H is made perspectively equal to the projection of the cornice $p\,s$, as before, and is best obtained from the section N″ S″; this being transferred to the plane N′ H, as will be easily understood from the figure; the quantity each raking moulding of the pediment projects, is equal to that of the same moulding where horizontal. By these means the perspective image of the oblique section made by the plane passing through $n\,n$ (*fig.* 4.) is obtained, and then the mouldings are drawn to the vanishing point through the various points of it, and the line H W′ will cut S″ W in the proper point corresponding to w (*fig.* 4.)

The modillions of the cornice are best determined on drawings made to small scales, by a plan such as that employed in *fig.* 1, and their perspective forms drawn by eye; but for the purpose of further illustration of this subject, the mode of finding them will be shown when no plan is made use of.

Draw A B, the intersecting line of the plane of the *soffite* of the *corona*, (see *figs.* 4 and 5.) through the proper point x in L P, at right angles, of course, to L P*, and draw $x\,y$ to the vanishing point; produce the line corresponding to the point Z in *fig.* 4 to Z in $x\,y$, and transfer Z to 1 in A B by the principle explained, (33, p. 145.) Set off the widths and intervals of the modillions, as given by the geometrical plan and profile *figs.* 5 and 4, along A B, and transfer them to $x\,y$ by means of the same point by which z was transferred to 1; then from the points 2, 3, 4, 5, 6, &c., in $x\,y$, thus found, draw by means of the vanishing point, on the perspective soffite, the lines of the tops of the modillions corresponding to 2, 3, 4, &c., in *fig*. 4. The ogee moulding which runs round them, and the inner angle of the soffite, must be drawn by eye, or its various mitre sections may be determined in the same manner as those of the other mouldings of the cornice, if the scale admits of or requires it. The upright lines at the backs of the modillions are determined, as will be seen in the figure, either from the detached plan, if one be used, or by means of the intersections of the perspective lines of the tops of the modillions on the soffite, and these will be all that can be usefully obtained by geometrical constructions even when the drawing is very large; the rest can only be put in by hand.

---

* Because the soffite of the corona being a plane parallel to the ground, its intersecting line will be parallel to the vanishing lines of such planes, or parallel to Y Z. (See *fig.* 2.)

The same proceeding is to be adopted for the modillions on the other parts, but those on the fronts D M and M F are not shown, to avoid confusing the more essential lines.

In drawing in the curved lines of the various mouldings of such a cornice, the effects of perspective must be borne in mind, and the curves delineated accordingly; this renders it necessary that the subject of the projection of curves be now entered on.

§ OF THE PERSPECTIVE PROJECTION OF THE CIRCLE AND OTHER CURVES.

The projection of a right line has been shown to be always a right line, because it is the intersection of two planes; and therefore it is determined, if two points in it are found; which may be either the intersecting and vanishing points, or the projections of any two points of the original. But if the original line be a curve, the projections of two points are not sufficient to determine its image, and a new set of principles is required.

The projection of the most important of all curve lines, that of by far the most frequent occurrence, the *circle*, may be determined, in many cases, more readily than by finding the projections of any number of points lying in the periphery.

The rays proceeding from every point of the circumference of a circle to the vertex, form a *cone*, provided the vertex is not situated in the plane of the circle*; and the section of this cone by a plane of projection must in every case be a *conic section*, which will be the *projection* of the original circle.

If the circumference of an original circle do not *touch*, or *cut*, the *station line* of its plane, (Def. 6, p. 125,) the projection will be either a *circle* or an *ellipse*;—a *circle*, if the plane of the curve be parallel to the plane of projection, that is, if the section be made parallel to the base of the cone, or if the section be subcontrary (Geom. App. Pr. III.): an *ellipse*, if these planes are oblique to each other, or if the section be not subcontrary.

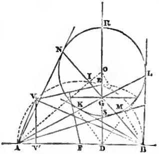
If the circumference *touch* the station line, that is, if the vertical plane be tangential to the surface of the cone, then the curve of the projection is a parabola, because the plane of projection which produces the section is in that case parallel to a side of the cone, or to the *vertical*, (Def. 15, p. 134.) of the point of contact.

If the circumference *cut* the station line, that is, if the vertical plane cut the cone, the projection is an *hyperbola*; and in this case the projections of the two parts of the circle lying on each side of the station line are opposite hyperbolas. (Geom. App. passim.)

Thus then in every position in which a circle may be situated with respect to the plane of projection, its image is a conic section, the data for describing which the principles of linear perspective enable us to find with great ease.

---

* It is almost needless to observe, that if the vertex be in the plane of the original circle, the projection of this will be a right line. (2, p. 126.)



Let O be the centre of the original circle, A B the *station line* of its plane, and let V′ be the point where a *vertical* (Def. 15, p. 134.) perpendicular to the station line meets it; the diagram representing the plane of the circle, and not the plane of projection. (See p. 154, and 31, p. 142.)

From this position of the station line, it is immediately seen that the projection of the circle must be an ellipse, wherever the plane of projection may be assumed; and as this plane is in every position parallel to the same given vertical plane, all the elliptic sections which can be made of the cone of rays will be *similar*.

Draw the diameter R S perpendicular to A B, and produced to meet it in D; draw the tangents from D, and the chord joining the points of contact, cutting R D in G; and make D E equal to the tangent from D. Draw V′ V perpendicular to A B and equal to the length of the vertical, so that V represents the vertex brought into the original plane; bisect V E by a perpendicular, produced to cut A B in F. On F as a centre, with F V or F E for a radius, describe a semicircle, cutting A B in A and B; or make F A, F B each equal to F V and F E. From A and B draw lines through G, cutting the circle in K, L, M, N; the two chords K L, M N, will be the *originals* of the *axes* of the ellipse, that is, the images *k l*, *m n*,[*] will be axes of the ellipse, which is the projection of the original circle, and therefore G will be the original of the centre of the ellipse.

For M and N will be the points of contact of the tangents from A, as K and L will be the points of contact of the tangents from B, (Geom. App. Lemma to Prob. 16.) and since A L, B N are *harmonically* divided by K and G, and M and G, the projections of K L. M N will be *geometrically* bisected by the image of G, (51, p. 153.) therefore the images of M N, K L, will be *diameters* of the conic section, which is the projection of the circle; and *g*[*] will be its centre.

And since A L, A N, A M meet in a point A in the station line, as do also B N, B L, B K, the projections of these lines will be parallel, (29. p. 134.) that is, the projections of A N, A M will be parallel to the diameter *k l* of the ellipse; and those of B K, B L will be parallel to the diameter *m n*, (Geom. App. 1. Pr. 6.) consequently these diameters will be *conjugate* to each other,

---

[*] See notation employed, note p. 126.

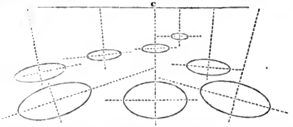
(Geom. App. Schol. Prop. 19.) And from the semicircle on A B, passing through V V E, the angle A V B is a right angle: these conjugate diameters, which are parallel to A V, B V, (29, p. 134.) are therefore perpendicular to each other, and must consequently be the *axes*. The principle of this construction will be apparent by considering, that it is not only necessary that the *verticals* A V, V B shall be at right angles, but that they be drawn to points A, B in the station line, so that A L, B N, drawn from these points through G, may be the originals of conjugate diameters.

From this construction and demonstration, it is clear that while the original circle and the station line remain the same, the points G and E will be the same, wherever V′ may be in that line, and whatever may be the distance of the vertex from the original plane, or the length of V V′; and that also the construction is quite independent of the angle of inclination of the vertical plane, or of the plane of projection, to the original plane, and that, therefore, whatever that angle may be, K L, M N will be the *originals* of the axes, and these axes themselves will be parallel to V A, V B.

If V′ had been at D, or in the station point of the diameter perpendicular to the station line, it is obvious from the construction that the chord through G, and the diameter R S, would be the originals of the axes; and these would, therefore, be *parallel* and *perpendicular* to the *intersecting line* of the original plane. But when V′ is on one side of D, it is clear that the axes of the ellipse will be *inclined* to the intersecting line; and of the two points A, B, that which is farthest from V′ is the station point of the original of the *major axis;* and the farther V′ is from D, or the farther the original circle is from the vertex, while its centre is at the same distance from the station line, the more inclined the axes will be to the intersecting line. And also if two original circles were situated one on each side of V′, but at equal distances from it, their centres being in a line parallel to the station line, then the axes of the ellipses representing them would be inclined in equal angles respectively to the intersecting line, but in opposite directions.

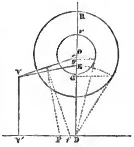
If the original circle be supposed to be farther from the station line, its centre being still in D O, the point G will be nearer the centre O, and the centre of the ellipse will approach nearer the perspective centre of the circle. The line V E will in this case form a more obtuse angle with A B, and the centre F of the semicircle will be farther from V′: the semicircle A E B will be larger, and the inequality between V′ A and V′ B will be increased; so that B N will form a smaller, and A L a larger angle than they do in the present example; hence the axes of the ellipse will be more nearly parallel and perpendicular to the intersecting line.

In the annexed figure, *equal* circles are represented in different situations, and at different distances with reference to the



intersecting and station lines, and the inclinations of their axes as determined on the foregoing principles are shown, C being the centre of the vanishing line of their common plane.

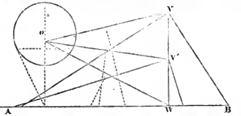
If two or more circles are concentric, the same point will not be the common original of the centres of the ellipses into which they are projected. When the vertex is at any finite distance, it will be found that the smaller the circle, the nearer the real centre will the point G be, through which the chords of the tangents pass: hence, as will be seen by the figure, the centres F, *f*, &c., of the semicircle by which the station points of the originals of the axes are determined, will vary; *f* being farther from V′ for each successive interior circle, consequently the axes of the ellipses representing an inner circle will be inclined to the intersecting line in different angles from those in which the axes of the exterior circle are inclined, except in the case when the centres of the circles are in the line from V′ perpendicular to A B; and then the axes of all the ellipses will be parallel and perpendicular to the intersecting line, as was before mentioned, though they will have different centres.



The centres of the ellipses which are the projections of such concentric circles will lie in a line drawn to the centre of the vanishing line; because the originals of those centres, G, *g*, &c., lie in a line, perpendicular to the station and intersecting lines, (25. p. 131).

The change produced in the inclination of the axes of the ellipses representing circles lying in different planes must be next considered.

Let *o* be the centre of the circle, and the base of a right cylinder, standing on the original plane, of which A B is the

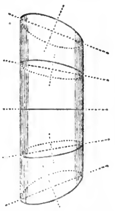


station line, the vertical plane being supposed perpendicular to it. Let the cylinder be supposed to be cut by planes parallel to the first plane: the *station lines* of these planes will be parallel to A B, and all circular sections of the cylinder will be at equal distances from their station lines: but the length of the *vertical* perpendicular to each station line, will vary according to the distance of the cutting plane from the common vanishing plane, or according to its distance from the vertex.

Thus if V W were the vertical to the plane (A B), and the next section of the cylinder were made by a plane at a distance from (A B) equal to V V', the *vertical* to the second plane would be equal to V' W; and so on, if more sections were supposed to be made. But it will be seen by referring to the construction p. 197, that the position of the originals of the axes of the elliptic projections depends on the length of the vertical; and that, therefore, these originals will vary for each successive section. The axes of the projection of the section made by the plane (A B) will be more inclined to the station line than those of the projection of a section made by a plane nearer to the vertex, or to the vanishing plane. If a sectional plane be supposed to pass through the vertex, V and W will coincide, or there would be no *vertical.* The projection would of course be a right line (2, p. 126), and the axes may be supposed parallel and perpendicular to the vanishing line.

If two sections of the cylinder were made by planes equally distant from the vertex on contrary sides of it, their *verticals* will be equal, and the originals of the axes will be inclined in equal angles respectively to the station line of each section; but these axes will incline in *contrary* directions to the vanishing line, though in equal angles to it.

The adjoining figure is the projection of the cylinder just referred to; and the ellipses formed by the sections of parallel planes are shown as deduced from the data; the axes of the ellipses on the other side of the vanishing line, but formed by planes at the same distances from their common vanishing plane, being equally inclined to that vanishing line, but in contrary directions.



It is hardly necessary to observe that these principles apply equally, whatever may be the absolute position in space of the cylinder, whether this be supposed vertical, horizontal, or oblique.

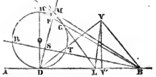
If a plane passing through the axis of the cylinder and the vertex were perpendicular to the vertical plane, then it is obvious that the axes of the ellipses resulting from the parallel sections would be parallel and perpendicular to the intersecting line; but when the cylinder is on either side of the vertex, then of the ellipses representing the parallel sections, those that are nearer the vanishing line of their planes, will have their axes more nearly parallel and perpendicular to that vanishing line, than the axes of the more remote sections, whether above or below; an analogous effect being produced by the proximity of the plane of the section to its vanishing plane, as was produced by the increasing distance of the equal circles lying in the same plane. (See *fig.* p. 199.)

If the cylinder were *oblique*, or if its axis were not parallel to the vertical plane, the constructions for finding the originals of the axes of the ellipses would be rather more complicated, because at every different section the distance of the station line from the centre of the circle would vary, but the general principles would apply equally, and no particular example is necessary, for reasons which will be hereafter given.

Although, for the reasons just referred to, it is not often necessary in practice to find the axes of the elliptic projection of an original circle, yet the student should make himself perfectly master of all the general principles just explained, so that he may always be able to decide, from the relative situation of any circle to the vertex and plane of projection, (or vertical plane,) what will be the direction of the axes of the elliptic projection; for of all errors in drawing, none are so fatal as those on this point, and yet none are so common, from want of an accurate knowledge of the subject.

Every artist, as soon as he makes an attempt to draw, although he may be unacquainted with the geometrical principles of linear perspective, perceives the apparent convergence of lines really parallel, and is soon master of the general modifications of forms of original objects, occasioned by viewing them from different points; but very few observe so accurately what they *see* as to draw a circle in perspective correctly, and hardly any draw it with a correct direction of the axes of the ellipse, unless they are conversant with the geometrical rules of the art.

One very flagrant error committed by artists, which the student should be especially warned against, is in the delineation of cylinders with their circular sections, or of the lines produced by such imaginary sections on their surfaces. Instead of drawing these as correct ellipses, with the outline of the cylinder as tangents to the curve, they almost invariably design the circular section, seen obliquely, as the segment of a circle, and the exterior lines of the cylinder, consequently forming an angle with the curve, instead of flowing gracefully into one another, as they necessarily do in nature; and thus, from ignorance, lose the advantage in their pictures of a really graceful form, which that of the ellipse so decidedly is: this will be further explained subsequently.



If the *station line* A B of the plane of an original circle touch the curve, the *vertical plane* is tangential to the cone of rays, and consequently the projection of the circle will be a *parabola*, (Geom. App.) as has been stated, (p. 196.) Draw a diameter perpendicular to A B, cutting it in D, and join V D; draw V B perpendicular to V D, cutting A B in B; from B, with B D for a radius, cut the circumference of the circle in E, and draw a line through D and E. Bisect the angle D V B by V L, cutting A B in L; and from L, with L D for a radius, cut the circumference in G; through B and G draw a line to cut D E in F, and through L and G draw another to cut D E in M. Then D E will be the original of the axis, F that of the focus, B M will be the original of the directrix, and F G will be the original of the ordinate through the focus, or the parameter of the axis. (See 5, p. 77.)

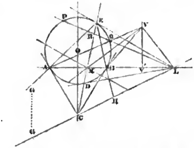
For B D being a tangent to the circle, B E, its equal, will also be a tangent to it, and therefore D E is the chord of the tangents from B, consequently B H, or any other line B R, will be harmonically divided in F and G, or S and T, (Geom. App. Lem. Prop. 16.) therefore *f h*, *f g*, *r s*, *s t*,* will be equal. (46.) The angle D V B of the verticals (Def. 15.) of *d e*, and the lines through B being a right angle; the projections of these last lines will be perpendicular to that of D E, (28.) and will be parallel among themselves, (29.) consequently *d e* must be the axes, *e* being the vertex, and R T, H G are the originals of double ordinates to it. (Geom. App.)

---

* That is, the perspective projections of F H, F G; R S, S T.

Again, from the equal angles D V L, B V L, each being half a right angle, the projection of the triangle M F G will be isosceles, and *m f* will be equal to *g f*: consequently B M will be the original of the directrix, and F will be the original of the focus; the line *l m* being a tangent to the parabola at *g*.

If the station line A B cut an original circle, the projections of the two segments will be opposite hyperbolas, as has been stated, (p. 196.) Draw a diameter perpendicular to A B; draw the tangents to the circle at A and B, meeting in a point C in that diameter. Join V A, V B, and bisect A V B by V M, cutting A B in M; draw a line through C and M, cutting the circle in D and E. Draw V L at right angles to V M, and through L and C draw a line; draw E A, E B produced to cut L C in G and H. Then C is the original of the centre of the opposite hyperbolas; C E, C D* are the originals of the semi-major axis;



---

* The *projection* of any *finite* right line which cuts the vertical plane will be *infinite*; for the point of intersection having its projection at an infinite distance, the projection of the original must be continued *ad infinitum* to that infinitely distant point. Thus the true projection of the line A B (see fig. p. 127) is not the finite segment *a b*, but the infinite line *a f*, *minus* the segment *a b*.

Consequently, in the present instance, the line C E, which cuts the vertical plane in M, will have an infinite projection; and the semi-major axis *c e*, is not the true projection of that line, but is the *complement*, as it may be called, of the true projection: or *c e* is the finite segment which unites the two infinite segments which are the true projections of the parts C M, M. E; and is analogous to the segment *a b*, of the figure above referred to.

This leads to another circumstance respecting the hyperbolas, considered as the projection of a circle, which is worth remarking. When these curves are regarded in this light, it is the two uninclosed portions of surface, lying on the concave side of the curves, and *not* the space between the two curves, which represent the two portions of the original circular area lying on each side of the station line.

D and E are the originals of the vertices, and any line through L will be the original of an ordinate ; C A, C B are the originals of the asymptotes. H G is the original of the conjugate axis, and G and H are the vertices of it. Any line through M will be the original of an ordinate to the conjugate.

For A B being the chord of the tangents from C, by construction, C E is harmonically divided in D and M, (51.) and consequently $c e$, $c d$, the projections of C E, C D, will be equal. Again, because the angle A V B is bisected by V M, and V L is drawn at right angles to it, V A, V M, V B, V L, are harmonical lines, (Geom. II. Prop. 50, Cor. 1.) and therefore A L is harmonically divided in M and B ; consequently C E will be the chord of the tangents from L, and therefore the tangents at D and E meeting in L, a point in the station line, their projections will be parallel, and the image of any line P Q drawn through L will be parallel to them, and will be bisected by the image of D E, (51, p. 153.) From the right angle M V L, these projections will be at right angles to the image of D E, therefore this last is the axis, the points d and e are the vertices of the opposite hyperbolas, and C is the original of the centre. Again, since the points A and B can have no images, the projections of the tangents C A, C B will be asymptotes to the curve ; and L C is the original of the conjugate axis, since its projection is parallel to the tangents at the vertices of the transverse axis. The projections of E G, E H being parallel to the asymptotes $c a$, $c b$, (29.) and drawn from a vertex of the major axis, G and H will be the originals of the vertices of the conjugate. Again, any lines drawn through M will have their projections at right angles to this conjugate axis, or parallel to the transverse, and such lines will therefore be ordinates to the former.

The projection of any conic section, taken as an original curve, will always be a conic section, because, since that original curve must be the section of a conic surface generated from a circular base, situated somewhere, and somehow inclined to the plane of the curve, the projection of this curve is only a new section of the same conical surface, and therefore must be a conic section.

If the proposed original curve be an *ellipse* not touching or cutting the station line of its plane, its projection will always be an *ellipse* or a *circle*, and it will be a parabola or a hyperbola in the other suppositions.

If the original curve be a *parabola*, neither touching nor cutting the vertical plane, its projection will always be an ellipse, or a circle touching the vanishing line of its plane.

For if the circular base of the cone of rays *touch* the vanishing plane, this plane, since it passes through the *vertex*, or apex, of that cone of rays, must be tangential to the conical surface ;

and the section of this surface by the original plane, which is parallel to that vanishing plane, will be a parabola, namely, the given curve. If it should happen that the circular base is parallel to the plane of projection, the image of the parabola will be a *circle;* but if the plane of the base and the plane of projection be not parallel, the image must be an *ellipse.*

If the original parabola *touch* the station line, its image will be a parabola *touching* the vanishing line.

For in this case the circular base must touch the vanishing line, or the section on the parallel original plane would not be a parabola; and as the point in which this touches the station line can have no projection, the image must be a parabola.

If the original curve be opposite hyperbolas, one lying on each side of the station line, their image will be a circle or ellipse, cutting the vanishing line, according as the circular base is parallel or oblique to the plane of projection. If the station line be an asymptote to the curve, the image will be a parabola cutting the vanishing line in one point.

If one of the original hyperbolas *cuts* the station line in one point only, both must cut it in a point; the images of the curves will be opposite hyperbolas, each cutting the vanishing line in one point[*].

If one of the original curves touches the station line, the other cannot touch or cut it at all; in this case the image will be a *parabola,* cutting the vanishing line in two points.

If one of the original curves does not cut the station line at all, and the other cuts it, it will cut it in two points; the images in this case will be opposite hyperbolas, one of which will cut the vanishing line in two points.

The foregoing theorems on the images of the parabola and hyperbolas are not purely hypothetical, though they are never required in the delineation of *objects;* but in the projection of shadows, and still more in the application of geometry to the projection of the *reflections* of objects as seen in polished surfaces, they may occur.

Thus if a perspective drawing of the interior of a room were made, with a mirror, inclined to one of the walls, and also to the plane of projection, represented in it, some parts of the chamber, and of objects in it, would be seen reflected in the glass. The mode of finding the projections of these reflections may be analysed into finding the projection of the reflection of one point. Now the reflection will be in the intersection of a plane with the mirror, passing through the original point, and the vertex, and perpendicular to the reflecting surface; and if an ellipse, or other conic section according to circumstances, were supposed in space, having those two points for its foci, and the intersection for a tangent, the tangent point will be that reflection of the original point; the projection of such nn ellipse would only be a right line, because its plane passes through the vertex; but if the curve were supposed turned down on the intersection, till it coincided with the plane of the mirror, the tangential point will remain the same, and then the curve may be projected, in order to find that point; and the curve of the projected ellipse, or other section, may be a parabola or hyperbolas, according to the foregoing theorems.

---

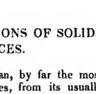
[*] Two parallel right lines of sufficient length have been shown to *appear* as curved lines gradually approaching each other. (See p. 138.) If two opposite hyperbolas were drawn on a flat wall, and the eye were placed in the vertex of the right double cone of which they were the correct section by the plane of the wall, these curves would *appear* as two parallel right lines; because parallel ordinates intercepted by the curves would every where subtend the same angle at the eye; that produced by the section of the conic surfaces by a plane passing through the vertex, and therefore the ordinates would appear equal. This is the converse of the former optical delusion.

As such problems are rather ingenious exercises, than of much *utility*, it does not enter into the plan of this work to give any examples; but this statement was made to explain the reason for giving the theorems alluded to, which might otherwise have appeared entirely superfluous.

It must not, however, be supposed that it is always necessary to adopt this complicated construction, for the projection of the reflection can be found, of course, much more easily by right lines only; but on some occasions, the conic sections may be employed, and the foregoing theorems will afford an useful exercise to the mind of a student desirous of learning the subject systematically.

§ ON THE PERSPECTIVE PROJECTIONS OF SOLIDS WITH CURVED SURFACES.

The cylinder is, to the practical draughtsman, by far the most important of all solids with curved surfaces, from its usually forming a constituent part of most works of human art, whether buildings or machines, &c. Before proceeding, therefore, to any further examples of the mode of delineating such objects in perspective, it will be necessary to explain some general principles relating to the projections of the cone and cylinder, considered simply as geometrical solids.



Let a line Y X be supposed to pass through the vertex V, parallel to the axis of a cylinder, whether right or oblique, and to cut the plane of the circular base, or that of any section parallel to it, in X or Y; from either point draw two tangents to the circle, as X R, X S, or Y R, Y S; and from the points of contact, R, S, let two lines be supposed drawn on the solid, parallel to its axis, and therefore representing the *generator*[*] in two of its positions. It is obvious that two planes passing through V X, and X R or X S, will be tangential to the cylindrical surface in the lines R R, S S; and therefore all rays from the vertex, lying in these planes, will also be tangential to that surface in those lines: hence these two lines, R R, S S will be the originals of the *contour*, or limiting outline of the solid. It is clear that the images of the tangents X R, X S will be in one line with those of R R, S S, (2, p. 126) and will pass through the *image* of the point X or Y. V X is, moreover, the radial of the parallels R R, S S, and of the axis of the solid; and therefore will, by its intersection with the plane of projection determine the vanisbing point of these parallels: hence it appears that the image of the axis, and the outlines of the cylindrical surface, will have a common vanishing point, provided the axis of the solid is not parallel to the plane of projection, and that vanishing point is the *image* of X.

If the axis of the cylinder be parallel to the plane of projection, V X will be parallel to it, and the lines R R, S S will have no vanishing point, or their images will be parallel: in this case V X will be the vertical (Def. 15, 134, and 29, p. 134) of X R, X S, and the images of these lines will also be parallel, and in the same right lines with those of R R, S S as before.

---

[*] Conical and cylindrical surfaces are conceived as produced by the motion of a right line, called the *generator*, always passing through the curve of the base; and either passing through a point, or moving parallel to a given fixed right line.

X would in this case he in the station line of the plane of the circle, hence the chord R S would be the original of a *diameter* of the elliptic projection, and therefore the images of R R X, S S X, would be tangents at the vertices of that *diameter*. But R S would only be the original of the major axis, on the supposition of R S being parallel to the intersecting and station lines of the plane of the circle, (p. 198.) or on the supposition of the cylinder being a right one, and having its axis in a plane passing through the vertex, perpendicular to the plane of projection.

In every case the images of R R X, S S X, will be tangents to all curves, which are projections of sections of the solid by planes, cutting it in any direction*. For since the original curves of such sections must lie wholly in the cylindrical surface, they must *touch* R R, S S, in a point of each, and therefore the images of R R, S S must *touch* the projection of the section in that of these points; and they cannot *cut* that projection, or meet it in more than one point; because in such a supposition, part of the projected curve would lie beyond these images, which cannot be the case, since these images form the visible limits of the solid.

Hence if the axis of the cylinder be *oblique* to the plane of projection, the images of R R, S S cannot be tangents at the extremities of a *diameter* of an elliptic projection of a section, because these images cannot be parallel.

And from these theorems it immediately appears, that when the vertex is at a finite distance, and not within the cylindrical surface, *less* than half of the surface of a solid cylinder can only be seen at once.

If the radial V X fell within the cylindrical surface, no tangents could be drawn from X to the circular base, and there could be no *right lines* R R, S S, drawn on the cylinder for the originals of the outline. In this case, therefore, this outline would be the image of any section which terminated the solid nearest to the *vertex;* and the image of the end of the solid farthest from the vertex would be seen wholly within the former.

A *cylinder* may be considered as a *cone* having its vertex at an infinite distance (see note, preceding page); V X may, therefore, be supposed to pass through such an infinitely distant vertex, and the solid may be regarded as a cone. A practical rule for finding the original of the outline of a cone may be deduced from what has preceded relative to the cylinder.

If a line V X be drawn through the vertex, or point of sight, and through the apex † of a cone, and produced to cut the plane of its circular base in a point X, then tangents to the circle drawn from X will give two points in its circumference, from which lines being drawn to the apex of the solid, they will be the originals of the outlines of its projection.

---

* Of course sections by planes parallel to the axis are not intended here, for these would be parallel right lines.

† The term apex is used here, to obviate the confusion arising from the same term, vertex, being applied to the point of sight, and to the vertex of the solid.

If the perpendicular height of the point of sight be equal to that of the apex of the solid above the plane of its base, then V X will be parallel to that base and therefore will not cut it, or the point X may be considered as infinitely distant; and the tangents to the circle will be parallel to V X, and will therefore be at the extremities of a diameter; consequently, in this supposition, the originals of the visible outlines will bisect the solid, and just half the conical surface will be visible.

If V X cut the plane of the base *beyond* the vertex, or so that the solid is between this point of sight and the point X, then more than half of the conical surface will be seen. This happens when the perpendicular height of the eye is *greate*r than that of the apex of the solid above the plane of its base.

But if the height of the eye be *less* than that of the apex of the cone, then the point of sight is between the point X and the solid, and less than half of the surface will be visible. If the point of sight were *within* the conical surface produced, the point X would fall *within* the circular base, and no tangents could be drawn to it, hence as in the corresponding case of a cylinder there could be no *straight outline* to the projection of the solid, but this outline would be the projection of any section of the conical surface, terminating the solid, and the projection of the apex would fall within it.

If the vertex V be in the conical surface, the projection of the apex will coincide with the point in the base, where the generator passing through V meets it. And if the vertical plane pass through this generator, the whole solid will be projected into a parabola, the apex will have no projection. These are hypothetical cases which cannot be realized in nature.

But if the vertex V be on the surface of a right cylinder the effect may be observed in nature: when a person stands in the doorway of a rotunda, as that of the Pantheon at Rome, he will see the outlines of the pavement, and of the cornices, &c., of the cylindrical part of the edifice, as parabolas, while the lines of the panels in the dome will appear as ellipses; but to perceive this, it is necessary that the eye be held steady, and directed always to the same point in space*.

---

* In the present case, and in those formerly alluded to (p. 138), of a line really straight *appearing* as a curve, the modifications in the apparent forms of objects do not depend solely on geometrical principles; the laws of vision and the structure of the eye must evidently concur with these to produce the result. In this work of course these last considerations are entirely omitted.

Since any regular polygon can be inscribed within, or circumscribed round, any circle; and since the perspective projection of such a right line figure is very easily obtained, the usual practical mode of finding the projection of a circle is to find that of some such polygon; and the several points of this projection will, of course, be in the periphery of the ellipse required, which is then drawn through them by hand. As a few points are sufficient for this purpose, the circumscribing *square* is generally made use of; the perspective central points of the four sides of the image of this where they touch the circumference, and the projections of the four points in which the circle *cuts* the diagonals of the square, which are easily obtained, furnish a sufficient number of points in the curve.

This auxiliary square may be assumed in any position, and therefore, for the sake of facility of construction, a side is usually taken parallel to the plane of projection; consequently, the centre of the vanishing line is the vanishing point of the two sides lying in planes at right angles to this plane, and the principal radial set off from the centre each way along the vanishing line, gives the vanishing points of the diagonals (42.)

Let A B F G (Pl. 7, Fig. 1) be the assumed square, the original circle being supposed to touch the plane of projection; the diagonals of the square cut the circle in N, O, Q, R. The projection A B *f g* of the square being drawn, then the projections of these four points are obtained by means of the lines drawn to the vanishing point from T and U, and intersecting the perspective diagonals A *g*, B *f*, as will be seen from the figure.

It is obviously unnecessary to draw the whole square and circle, for a semi-circle being described on A B, and the half square A D E B being completed, the radii D L, E L, will give the points N′ and O′, by which T and U may be obtained; that is, A T and B U must be made equal to the distance of N′ and O′ from A D, D E, B E. It will be also seen, that the semi-circle might have been described on *f g*, or on *d e*, or on any line which is the projection of one equal to A B, and parallel to the intersecting line of the plane of the circle; for the figure *f h i g* being constructed on such a line, it will give the points *w*, *x* corresponding to T and U obtained before, for reasons which will be presently explained.

If any number of original circles were concentric, the diagonals of the circumscribing squares would be common to all, provided the sides of these were parallel; and the points of each, corresponding to N and O, are readily obtained, as well as the projections of them, so that the projections of the ellipses representing these concentric circles are easily drawn.

The same principle is universally applicable to finding the pro-

Google

jection of any curved line; for the intersections of the perspective projections of right lines, passing through points of the original curve, will, of course, be points in the projection of that curve. If squares drawn on the plane of an irregular outline or drawing, as if for the purpose of copying it (see p. 56, *et seq.*) be put in perspective, the *trapezia* will serve as guides for drawing that irregular outline in perspective; thus in a view of the interior of a room, the outlines of pictures hanging or painted on the walls may be drawn in perspective. It is in this way that those toys called *anamorphoses* are made; the squares drawn on a figure or landscape are projected into distorted perspective, by assuming the vertex very near, and the plane of projection perpendicular to that of the original drawing; the outline of the figure, &c. is carefully copied in these irregular quadrilaterals, and this distorted copy, if viewed from a fixed point corresponding to that of the assumed vertex, will resume its correct natural appearance. (See p. 138.)

Examples will now be given for the purpose of showing the practical application of the principles relating to the elliptic projections of circles, and of the projections of cylinders, &c.

At Fig. 1, Pl. 4 is shown the geometrical profile of two bevelled cog-wheels, P is the apex of the cones of their teeth, this point being the intersection of the axles of the two wheels: from this section it will be seen that there are five principal circles **1, 2, 3, 4, 5**[*] in each wheel, lying in different planes and of different diameters, their common axles being perpendicular to the planes in each wheel.

In the following constructions the wheels are considered as isolated objects, and therefore the proceeding most applicable in such a supposition is adopted; but the learner will easily comprehend how this must be modified, if the wheels formed part of a machine, and if the planes of the circles of the larger wheel were not necessarily perpendicular to the plane of projection, as they are assumed to be in the present instance. A B (Fig. 2) is made equal to the diameter of the circle 3, of the largest wheel, in the intersecting line of its plane; C being the centre of the picture, and C X the distance of the vertex: on A B describe a semi-circle, draw the half circumscribing square A D E B, find A B *d e*, the projection of the whole square, and also those of the points in the perspective diagonals through which the ellipse will pass, and of the tangential points in the sides, or the perspective central points of those sides. Draw 1 C and the other lines to C from the points where 1, 2, 3, 4, and 5, cut P P:* produce de to m; then P P will be the intersecting line of a plane, perpendicular to the plane of projection and to the plane of the circle, and on this plane the perspective projection of the section through the centre of the wheel must be drawn. The points p', p', and p', which represent on this auxiliary plane those in the axle of the wheel to which the bevelled tops and ends of the cogs tend, must be found by drawing lines from P, P, in P P to C: transfer the points p' thus found to the perpendicular through l, the perspective centre of the square. The lines ap', mp', will determine the points T′, W′, u', &c. of the section on the plane (PP), and it was for the purpose of obtaining these points more accurately than could be done by the perspective section through L′ lp that that on (PP) was constructed; lines drawn parallel to A B, through T′ W′ u' m, &c., the various points of this section will be the sides of the *perspective* images of the squares circumscribed round the various circles of the wheel. The points o' l' in p' p', which represent the intersections of the planes of these circles and the axle of the wheel, must be transferred to pp; lines drawn through these points o, l, &c., to X and Y, will be the perspective diagonals of these circumscribing squares, and these diagonals will give the angles of the projected squares; thus the lines through o to X and Y, will give q, s, t, r, and the lines qr, ts, drawn to C, will be the other sides of the square round the circle 1 of the wheel.

---

[*] For the sake of simplifying the constructions as far as possible, the section of the wheels is not shown as it would really be: thus, for instance, the plane 3 of the upper face of the flanch is shown as coinciding with the outer edge of the cogs, and the lower face 4 with the bottom of the cogs, and so on, which might or might not be he case in the real wheel.

On qt describe a semi-circle, and proceed in the same way as before to find the projections of the points in which this circle cuts the diagonals of the square circumscribing it. These constructions must be repeated till the sides of the squares and the diagonals for all the circles of the wheel are found; and the points in them through which the elliptic projections of those circles will pass, or where they touch the sides.

As the correctness of the ellipses which are to be drawn in by hand will depend on that with which these points are found, every care must be taken to project the squares and their diagonals very accurately, and recourse should be had to some mode of verifying them; as for instance, by finding some of the angular points of the images of the squares by other constructions. This should particularly be done for such points as T, u, W, &c.; these will also be determined more correctly by making use of lines drawn from A, L, B, de, &c. to p, p: thus Ap, Bp, dp, and ep, will verify the points q, t, r, s, and so on.

In the figure, the projections of the eight sections of the wheel

---

* The line aC is to be drawn from any point, a, in A B, produced at pleasure, as far as convenient from the centre C of the picture; it need hardly be observed, that it is not necessary that the geometrical section should be in the place shown in the plate, but the points from P P must be set off on a line drawn through a perpendicular to A B, if it be not.

Google

by four planes passing through its axle, through the diagonals of the squares, and parallel to their sides are shown. Since the principal points of these sections are determined by the previous constructions relating to the projections of those squares and their diagonals, it is advisable to complete them; by so doing, not only will the various points through which the ellipses are to be drawn be kept more distinct, but the ellipses themselves may be drawn more correctly.

It will be seen from Fig. 1 that the plane of the largest circle 4 of the smaller wheel, does not touch the corresponding circle 3 of the larger wheel at *A*[*]. Attention being paid to this, the mode of finding the projection of the circumscribing squares of the circles of this smaller wheel will be the same as that just described for the others; it will, therefore, be unnecessary to enter into the details of these constructions.

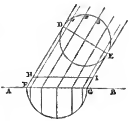
The next step necessary to drawing in the cogs, is to divide one ellipse of the projections of the wheels into the requisite number of *perspectively* equal parts. To do this, divide the semicircle on A B *geometrically*; and transfer the points of division to A B by means of ordinates perpendicular to this line; then lines drawn from the points thus formed in A B to C, will cut the ellipse in the perspective divisions. But it is obvious that the ellipse being drawn by hand, and therefore necessarily not perfectly correct, the perspective division of it into small quantities must be still more uncertain, and the points near the extremities of the longer axis will be altogether undefined, and must be determined by the judgment of the draughtsman: thus since there must be the same number of divisions in each equal portion of the real circle, he must take care to get in the same number of perspective points of division in each segment of the ellipse representing those equal portions of the circle, as for example between each diagonal of the projection of the square and the lines through the centres of the sides.

As, for the same reason, every precaution should be taken to get the principal points of perspectively equal divisions as correctly as possible, lines may be drawn from the points in one-half of the ellipse through its perspective centre, which will verify or correct those in the other half; this is shown in the plate, as done for the divisions of the smaller wheel.

To explain the principle of this perspective equal division of the projection of a circle, let D E be the diameter of one in

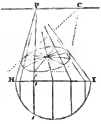
---

[*] This smaller wheel is assumed as having the planes of its circles parallel to the diagonal L D (*l d*) of the larger; this is only done to simplify the figure, and the same remarks will here apply as were made in the last note, and in that at the commencement of the example



an original plane; A B being its intersecting line. Let the circumference be divided into any number of equal parts in 1, 2, 3, &c., and let lines be supposed drawn through the points of division, perpendicular to D E, and produced to cut A B; F D, G E, being tangents to the circle. Now it is obvious that these parallels will cut F G and D E in the same proportional parts; and this last line is divided by them into the differences of the versed sines of the equal arcs. Therefore, if a semicircle be described on F G, or any line parallel to it as H I, and divided into the same number of equal parts with that on D E, the points in F G or H I may be determined directly from the divisions by means of parallel sines, as shown in the figure.

Now if the circle on F G or H I be supposed turned round on that line till the plane of it is parallel to the plane of projection, the points of division in the line F G or H I will not thereby be altered, and the circle will be projected into one similiar to the original, and consequently the *image* of F G or H I may be divided into the versed sines by means of the circle on that image; and the parallels F D, G E, &c. being projected into lines having a common vanishing point in the vanishing line of the plane of the circle on D E, the elliptic image of this will be perspectively divided by these converging lines.



Therefore, to divide an ellipse perspectively on this principle, take any point P in the vanishing line of the plane of the original circle, so that tangents P H, P I, drawn from it to the ellipse may touch this, as nearly as can be estimated, at the extremities of a chord parallel to the major axis. Draw a line parallel to the vanishing line to cut the tangents; and on H I, the intercepted segment, describe a semicircle which is to be divided into the proper number of equal parts; draw ordinates from the points of division perpendicular to H I, and from the points in this line where the ordinates meet it, draw lines to P, the assumed point: these will cut the ellipse in points which will divide it *perspectively* into equal parts.

If the divisions are required to commence from any point as 1 in the circle, a line P 1 must be drawn from the assumed point through the given one, and produced to cut H I in 1. Then the ordinate being drawn from I in H I, perpendicular to it, to cut the circle in 1, the equal divisions must begin from that point.

With this limitation it is most probable that the divisions of the circle may not fall into the ends of the diameter H I; and consequently, the divisions in the opposite semicircles may not lie in the same lines, having P for their vanishing point. In this case the complete circle must be drawn on H I, and the divisions of the opposite semicircles transferred by ordinates to the diameter, and the perspective divisions of the circumference of the ellipse must be determined from these: or the divisions of the perspective of the half ellipse being determined from the semicircle as before; those of the remainder of the curve may be obtained by drawing diameters through the perspective centre, as is shown in the Figure and in the Plate.

The point P may of course be taken any where in the vanishing line; but practically, when the directions for assuming it as above given are followed, the greatest number of the lines cut the ellipse least obliquely, and there are therefore only a few near its extremities which are very oblique to it.

In the example of the cog-wheel, the point C was made use of to divide the ellipse, only because it obviated the necessity of a new construction which would have confused the figure; but it would have been better to have taken one more consistent with the above condition, and in that case a new semicircle must have been described on the intercepted segment of A B.

Having obtained the divisions of the larger wheel, those of the smaller must begin from a fixed point, so that a cog of it shall fall into the space between two of those of the larger; hence a semicircle only being used, the mode last suggested, of determining the perspective divisions of the remainder of the circumference by perspective diameters, is employed as is shown in the figure.

Having got by these methods the divisions on the ellipse, the outlines of the cogs are to be drawn from them to p', p' and p' in the axle; but a great deal must be left to the eye and judgment of the draughtsman in so doing, for whatever care may have been taken, the divisions will not be sufficiently accurate to be implicitly trusted to, especially for those cogs which, being near the extremities of the major axis of the ellipse, are most foreshortened or most crowded. And attention must also be paid to the real form of the cogs, these being not *planes* on their sides, but curved surfaces, to facilitate the working of them in those of the other wheel, so that the upper face of the cog is narrower than the lower part, and this must be expressed in the drawing: the detached figure 3 on the plate showing four or five cogs to a larger scale, will explain the mode in which this is to be done.

The projections of the spokes of the wheel are most correctly obtained by finding the vanishing points of the *central lines* of their upper faces, the radials being drawn to make the proper equal angles at the vertex. But as the spokes taper from the centre, these vanishing points will not serve for their sides: to obviate the necessity of finding the vanishing points for the sides, the following plan is resorted to. Draw the elliptic projection of the small circle to which the sides of the spokes are tangents, as found from the geometrical plan of the wheel; the widths of the spokes where they meet the flanch may be proportioned so as to be perspectively equal, either by eye from the divisions for the teeth, or they must be deduced from the plan in the same way as those divisions were: then the lines of the spokes on the upper face must be drawn as tangents to the small ellipse. The thickness of the spokes is given by the perspective thickness of the flanch, as expressed by the two ellipses of its inner edge, but these details are best drawn by eye.

The real axles of the wheels are not shown in the plate, the mode of drawing them requiring no particular directions but such as are general to all parts of machines.

Let C (Fig. 3, Pl. 7) be the centre of the picture, C V the distance of it, A B the intersecting line, and R the centre of the vanishing line of a plane, forming any proposed angle C R V, with the plane of projection (32, p. 143) in which plane is situated the circular base of a right cylinder of the diameter A B, the circumference touching the plane of projection. A B *d e* being drawn as the projection of the circumscribing square, the ellipse must be drawn, either by means of the diagonals A *e*, B *d*, as explained in the preceding example, or by finding its axes by the constructions given in p. 197; V R being of course the distance of the station line of the plane (A B) from the intersecting line; or, if more convenient, the major axis of the ellipse may be found by Prob. 60, p. 90. Find S, the vanishing point of lines perpendicular to the plane (A B) (54, p. 159); draw *c* S, the perspective axis of the cylinder, and lines touching the ellipse for the outlines of it (p. 208): *c h* being made *perspectively* equal to any proposed length, complete the projection of the circumscribing square at the other end of the solid, by means of the lines A S, B S, *d* S, *e* S, &c., and draw the elliptic projection of the circle at that extremity which is parallel to the plane (A B).

The distortion, shown by the great inequality of size and form of the rhombuses, and difference in the direction of the axes of these ellipses, representing equal and parallel circles, arises from the proximity of the vertex to the plane of projection, and serves to illustrate the theorem (p. 200) of this change of direction as produced by the different distances of the original planes of the circles from the vanishing plane; the axis of the upper ellipse being much less inclined to the vanishing line, than the axis of the ellipse representing the more distant circle.

Let K D be assumed, or found (p. 163) as the intersecting line of a plane, perpendicular to the plane of projection, and inclined in any proposed angle[*] to the plane (A B); C E parallel to D K will therefore be the vanishing line of this second plane: and if A B be produced to cut D K in D, then a line drawn from the point of intersection to E, the intersection of the two vanishing lines, will be the *projection* of the intersection of the two original planes.

In order to find the projection of the original *ellipse*, formed by the section of the cylinder by the plane (D K): draw A K, B K′, and L M, perpendicular to A B, meeting D K in K, K′, and M. Now it is clear that the intersections of the plane (D K) with the two planes (A K) (B K′) of the sides of the parallelopiped, inclosing the cylinder, and with the plane through its axis parallel to those sides, will be perpendicular to the plane of projection: draw K C, K′ C, and M C, therefore as the projections of these intersections; then the points *a*, *b*, *n*, *o*, in which the two first cut A S, B S, *d* S, *e* S, being joined, the trapezium *a b n o* will obviously be the projection of the parallelogram[†] formed by the section of the parallelopiped by the plane (D K)

If lines be drawn to S from the points in which the ellipse cuts the perspective diagonals of the square A *e*[‡], they will cut the diagonals of the trapezium *a o* in points through which the elliptic section will pass: and lines from the tangential points in the sides of the same square will give the tangential points in the sides of the trapezium: these eight points will be sufficient for drawing the curve by hand, or the axes of the ellipse may be found by the Problem before alluded to.

P Q *t u* is the projection of another square, parallel to the plane (D K), and forming one end of a rectangular parallelopiped inclosing a right cylinder, the axis of which is therefore parallel to the plane of projection: complete the projection of the solid as terminated by the plane (D K), and draw the inscribed ellipses by means of the diagonals, &c. as before.

The sides of the cylindrical surface bounding its visible part may be drawn tangents to the two ellipses (see p. 208); but if the precise points where these tangents meet the ellipses are required, the following construction will give them when the axis is parallel to the plane of projection, whether the cylinder be right or oblique. Produce the axis of the solid $y\,z$ to cut the curve in points l and m, and to cut the vanishing line of the plane of the circle in x': then $m\,l$ is the projection of a diameter of the circle, and x' is its vanishing point. Find the vanishing point Z* of lines perpendicular to the original of $l\,m$, and lying in the plane of the circle; bisect $l\,m$ *geometrically* in i: then a line to Z through i, will cut the ellipse in the points r, s, where the tangent outlines of the cylindrical surface touch it. For the original of $r\,s$ is the chord of the tangents to the original circle, from the station point of the diameter of which $l\,m$ is the image (see fig. p. 197). Now this station point is that where a vertical parallel to the axis of the solid meets the original plane, because this axis is parallel to the vertical plane (see p. 207); s and r will therefore be the projections of the points where the tangents touch the circle.

---

[*] The cylinder being a right one, the planes of the parallelopiped inclosing it, A B *d e* S, will be perpendicular to each other; the side B *e* S is therefore perpendicular to the plane (A B), and is consequently perpendicular to the plane of projection: B K drawn perpendicular to A B is the intersecting line of the plane B *e* S. Hence B K M is the angle of inclination of B *e* S to the plane (D K.)

[†] If the constructions have been accurately made, *b a*, *o n* will be found to meet, if produced in their vanishing point, in E C, as they ought to do; and this circumstance will be a means of verification.

[‡] The 'square A *e*' means that the original is a square: this and similar expressions adopted to avoid unnecessary repetitions, can occasion no embarrassment to an attentive learner.

If the axis $c\,h$, of the cylinder, be oblique to the plane of projection, S c must be produced to l, and x', and Z, the vanishing point of lines perpendicular to the original of $l\,m$, must be found as before. Find a third point i, to complete the harmonical division of S l in m and i (Plane Geom. Prob. 15, p. 37): then a line drawn through i to Z will cut the ellipse in r and s, the points of contact of the straight outline with the ellipse.

For S l being harmonically divided, the three *rays* through the points l, i, and m, and the radial of the vanishing point S, will be four harmonical lines (Geom. II. Prop. 49). Now S also represents a point in the original plane of the circle (p. 207), and the original of the line S l will consequently be divided harmonically in the originals of the points i and m; consequently, the original of $r\,s$, which is perpendicular to the original of $l\,m$, which is a diameter of the circle, c being the perspective centre, must be the chord of the tangents from the point S in the plane of the circle, and S being the intersection of a line parallel to the axis of the cylinder (see Fig. p. 207), the bounding lines of the cylinder will be lines drawn through S, and the extremities of the chord drawn through i to Z.

It will be seen that the constructions in these two cases, when the axis of the solid is parallel or oblique to the plane of projection, are the same in principle: for, in the former case, $m\,l$ is geometrically bisected in i, because the fourth point of the harmonical division is at an infinite distance.

In Fig. 4 of the same plate, two cylinders are drawn cutting one another, for the purpose of showing the manner of obtaining the projection of the curve of their mutual intersection. The axis L*m* is perpendicular to the plane of projection, and as the cylinder is a right one, the circular bases, and all sections parallel to them, will be represented by circles; the axis and the visible outline of the surface will therefore have the centre of the picture for their vanishing point; the other cylinder is also right, and its axis is parallel to the plane of projection: A B is the intersection of the plane of the base N O, with the plane of the base *d e f g*.

---

* The two vanishing points referred to as Z do not come in the plate.

Draw N A, L L′, &c. perpendicular to A B, and A C, B C, L′ C, L C, N C, and O C to the centre of the picture* from *s*′, *t*′, and *u*′; where L′ C cuts *d e*, *p q*, and *f g*, draw lines parallel to *p r*, to cut L C in *s*, *t*, and *u*: on these points describe circles to touch the outlines of the solid; these circles will be the sections of the solid by the planes of the parallelopiped inclosing the other cylinder, and by the plane parallel to two of them passing through the axis *p r*. Complete the projection *h i k l* of the square inscribed within the base; produce C *h*, C *k*, to cut A B in *x*′ and *y*′; draw lines from *x*′ and *y*′ parallel to *p r*, to cut the circle N O in *x* and *y*; draw *x c*, *y c*; the eight points, in which these lines are cut by others, drawn from *h*, *i*, *k*, and *l*, parallel to *p r*, will be in the curve of the intersection. Lines from *q* and the other tangential points of the base *p q* will cut the three circles on *s*, *t*, and *u* in six other points; and these fourteen will be sufficient to allow of the curve being drawn by hand, taking care to make it *touch* the outlines of the cylinders.

The principle of this construction, namely, that of finding the projection of lines on the two cylindrical surfaces, which are known to meet each other, because they lie in the same plane, and therefore to have those points of intersection in the mutual one of the surfaces, is applicable on all occasions when this curve is required to be projected, as will hereafter be seen in the delineation of shadows.

The mode of delineating the perspective projections of *interiors* of buildings, &c. differs in no respect, in point of principles, from that by which *exteriors* are drawn; for if geometrical solids are considered as composing the general forms of the structures, it is of course immaterial whether it be the *external* face of their sides, or the *internal* cavity and the internal face of their sides which is considered.

The principal difficulty attending the drawing of interiors arises from the choice of the situation for the vertex, or the place from whence the spectator's eye is supposed to view them. A person standing in a room can of course only see the whole of one wall; of the others, the floor and the ceiling, he cannot see more than the eye can embrace under an angle of 40° to 50° (see p. 175), if so much. In order therefore to show more than could really be seen, for the sake of giving a more comprehensive view of the place, the situation of the spectator is assumed out of the room, one side or more of which is supposed to be removed for the purpose. Of course, a perspective view of the chamber, delineated from such an artificial point of sight, can never be *true to nature;* and, consequently, all such views give an erroneous idea of the length of the room, for any person looking at such a drawing, being from habit aware how much the eye can embrace at once, concludes that the artist must have stood far removed from the nearest parts of the foreground, though within the apartment, not presuming that a wall is supposed to have been removed for his convenience.

---

* The centre of the picture C is not shown in the plate, and some of the lines of the construction named are not drawn.

If the chamber be rectangular in its plan, it is better to take the plane of projection *oblique* to the side opposite the eye, for if it be assumed as *parallel* to that side, the horizontal lines of the two sides at right angles to it vanish in the centre of the picture, and those of the first side are horizontal and parallel; and thus a formal, unpicturesque view is obtained, equally objectionable with, and analogous to, that of an exterior projected on a plane parallel to one front of an edifice, instead of oblique to two. (See p. 174.)

Pl. 1. Fig. 4 is the plan* of a staircase, and Pl. 6 represents the projection of it on a plane passing through the point D, vertical to the floor and to the horizontal planes, and cutting them in the line A B; V C being the distance of the vertex, the spectator looking through the archway on the first landing place from the station V.

The principal upright lines of the projection might, of course, be deduced from the plan, by drawing lines to V from the various points of it, as was done for the building, Figs. 1, 2, Pl. 3, and on the same principle; and many are so obtained, both on this and other occasions; but on account of the rays from the right hand wall forming very acute angles with it, many essential lines, if thus deduced in the general outline, would be ill determined, and might subsequently cause embarrassment: the following is therefore the better mode of proceeding at first, for getting in the principal outline.

Draw a line Z Y (Pl. 6) for the *horizontal line*, or vanishing line of the floor, &c.; mark the centre of the picture, C, in it as convenient; make C *a*, C *b* accurately equal to C A, C B on the plan, and draw lines through a, C, and b, carefully at right angles to the vanishing line: make C V equal to the distance of the vertex, and draw the radials V Y, V Z, of the sides of the plan, making the correct angles with C V; or set off the distances of these vanishing points as obtained from the plan along the horizontal line, but in either case care must be taken to get these vanishing points correctly.

---

* The plan is drawn in the plate to a smaller scale than that from which the projection was really deduced; but in the text it is alluded to as if drawn to the true scale. A geometrical section of the staircase must be supposed, as giving the dimensions not furnished by the plan.

Now the perpendiculars through a and b represent the intersections of the walls, with the plane of projection, and therefore the heights of the various parts must be set off of their true dimensions along these lines. Make a A, b B, in them therefore equal to the true height of the eye above the floor of the hall, and A A′, B B′ equal to that of the whole staircase from the floor to the central highest point of the arches of the groin; draw lines through A, A′, B, B′ to Z. Make A′e', B′e', A′f', B′f'*, on these lines, perspectively equal to A E, B E, A F, B F in the plan (33,'p. 145), and complete the general outline of the geometrical rectangular parallelopiped which represents the *cavity* of the staircase included between its walls.

If the lines $e e'$, $e e'$, had been first obtained from the plan only by means of V E, V E, a trifling error in their position would throw the outline of the part between the plane of projection and the vertex very much out of correct drawing, on account of the oblique intersections of $e f$ and B B′, &c.

The groined ceiling. Make A′ G, B′ G equal to half E E, the width of the staircase, because the vault is a half cylinder longitudinally. Complete the lower lines of the imaginary rectangular solid which contains the arches of the groin; draw the diagonals $e g$, $f g'$, &c., on all the vertical sides, and also those on the three horizontal planes, for the purpose of obtaining the *perspective* central lines of these various planes; the diagonals on the three latter $e'f'$, $g g'$, &c. will also be the plans on those planes of the intersections of the diagonal arches of the groins, represented in the plan by the lines E F, E F.

Draw the half square on E E (in the plan), and find the points l and m where the semicircle on E E cuts the semi-diagonals; draw $l p$, $m q$ parallel to E F, E F, cutting the diagonals in n, p; o, q; draw $n r$, $k s$, $p t$ parallel to E E, cutting E F in w, x, y. Make $x s$ equal to half E E, and $w r$, $y t$, each equal to $l z$ or $m z$: then a semi-ellipse, having E F for its major axis, $x s$ for its semi-conjugate, and necessarily passing through r and t, will be the section of the cross elliptic groin-cylinder, by the planes of the walls, laid down on the plane of the plan. For the planes passing through the diagonals E F ($e g' f g$) will cut the semi-cylindrical vault in two semi-ellipses, and the cylinder which is generated from these, and which forms the cross vault of the groin, will be cut by the planes of the walls E F ($e' f g' g$) in semi-ellipses. Now the lines of which $z\,n$, $z\,o$, &c. are the *plans*, are on the cylindrical surface and parallel to its axis, and they meet the semi-elliptic sections in points of which n, o, p, q are the *plans:* the lines, of which $o\,w$, $q\,y$, &c. are the plans, lie in the cross cylindrical surfaces, and proceed from the same points, and therefore cut the planes of the walls in the points r and t.

---

* The points f, f, and one f' do not fall within the plate, but the lines through them perpendicular to the ground sufficiently indicate the outline meant.

Make $g'w$, $g\,y$, *perspectively* equal to E w or F y; and $g'z$, $g'z$ equal to E z[*]: draw lines to Y through w and y; and lines to Z through z, these four lines will, if correctly drawn, intersect in the points n, o, p, q, in the perspective diagonals $g\,g'$: draw the perspective perpendiculars $p\,p'$, $y\,t$, $w\,r$, $o\,o'$, &c.; make $y\,t$, $w\,r$ equal to $z\,m$, &c., by setting the absolute length of these lines along A G, B G from G, and transferring it by Z to r, t, &c. Then by means of the two vanishing points, the points n', o', p', q, in the diagonal elliptic groin lines are obtained; and the curves must be drawn through them, tangential at k' to $f\,e'$, and tangential to the perpendicular lines $g\,f$, $g'\,e'$, &c. The elliptic *projections* of the semi-circular section on the wall E E, and of the semi-elliptic sections on the side walls, are drawn in the same way, the latter through the points $g\,s\,g'\,t\,r$, and tangential to the upright lines at the three first points[†].

The constructions for this purpose have been explained at length, because they are generally applicable to the delineation of all groined arches, &c., and therefore of frequent occurrence in architectural drawings.

The inevitable distortion of forms which always attends interior views, arising from the proximity of the point of view to the greater part of the originals, is sufficiently manifest in this example, the *spandrel* $g\,t\,s\,x\,y$ representing a portion of the wall equal to $g'\,r\,s\,x\,w'$.

The arch on the landing, through which the spectator looks, must be next drawn. The upright lines are either deduced from the plan, or are better obtained by making $g\,h$ perspectively equal to F H, i having been obtained before as the perspective central point of $g\,g$; draw $f\,i$, $f\,i$, meeting the perpendiculars $h\,h'$ in h': then lines from Y through $f'\,h'$ will give i', the vertices of the arches; and the point in $i\,h'$ through which the elliptic projection will pass, is obtained by a quadrant described from i as a centre on $i\,i'$ as a radius, as is shown in the figure. The inner arch nearer the spectator, and equal to that just described, is drawn in the same way; and requires no further notice, except to caution the artist to make his perspective arches true semi-ellipses, having their axes with the correct inclination, and tangential to the perpendicular sides, so that there may be no angle at the junction.

---

[*] As these points should be accurately determined, the perspective curves entirely depending on others related to them, the constructions should be repeated on the lines A e, B e, $e\,e$, and the former results thereby verified.

[†] If the draughtsman be skilful and accustomed to *observe* forms, he will have no difficulty in drawing the curves from these few points found in them; but otherwise it may be necessary to find others, which, if done, should be obtained for the more *convex* parts of the curves. The following method for furnishing this aid to his hand and eye is worth mentioning, as applicable on similar occasions where still more necessary. In the plan, find by the Probs. 54, 59, &c., pp. 83 and 88, the points in $x\,s$, x F; where a tangent to the ellipse at any assumed corresponding points in the two quarter ellipses would cut those lines, find the perspective projection of these points of intersection in the lines $x\,s$, $g\,g'$ on the drawing, and through them draw the perspective tangents, which the curve, as drawn by hand, must be made to touch. This will be shorter and more conducive to accuracy, than finding the points themselves through which the curve must pass: for in drawing a curve tangential to a *line*, this guides the curve, as it were, for some short distance on each side of the point of contact; whereas in drawing a curve through a point there is no such guide beyond it.

The stairs and landing places. Make A a', $a\,a''$, B b', $b\,b''$ each equal to the height from the floor of the hall to the upper side of the first landing; for from this, to the upper side of the second, is the same distance as the first, because there is the same number of steps in each flight of stairs. Divide $a'\,a''$, B b' into twelve equal parts, for the *risers*, and draw lines to Z through the points of division, the points l, m in A e, B e, corresponding to the plan of the first and last *square treads* being obtained; the segment $l\,m$ must be divided *perspectively* into nine equal parts, (this is best done by the method explained, p. 187,) then perpendiculars through these points of division will give the lines on the walls where the upright riser of each step cuts those planes; but on account of the projection of the skirting board, they will not be the *visible* lines of these sections, an allowance must be made accordingly, as is shown on the drawing. This may be done by eye, or in the way which will be presently explained in describing the mode of drawing in the impost mouldings and plinths, &c. The lines of the edges of the steps are then drawn from Y through these sectional lines on the planes of the walls. The two remaining spaces of equal division of $a'\,a''$, B b' are for the risers of the winding steps, and the intersections on the walls of the risers of these four steps may be deduced from the plan.

It will be better to put the plan of the steps, and of the *well*, with its semi-circular termination, in perspective; either on the plane of the hall floor, at D, or on another plane parallel to it, the intersecting line of which may be taken any where at pleasure, as at d, (see p. 183, and Pl. 2.) From this perspective plan the ends of the steps next the well, and of the balusters, &c. are readily and accurately obtained. The lines of the four winding steps may either be drawn to their proper vanishing points, as found by radials constructed expressly; or, more shortly, by finding the points where they cut the line D d, which is easily done, this line being in the plane of projection, This proceeding will be understood from the figure, and from the plan. The radial Z V being set off along the vanishing line, and the angle of inclination of the plane touching the steps being constructed at the vertex thus brought into the plane of projection, the radials of these inclined planes will cut the vanishing line W X, of the side walls, in W and X, the vanishing points of the raking plinth, or skirting board, which may be drawn through them. If an imaginary plane, passing through the line D *d*, and therefore having that line for its intersecting line, be supposed perpendicular to the side walls, it will cut the impost moulding running round the staircase under the groin, and the skirting board, in *profiles* of those mouldings, &c.; these profiles may be drawn on such a plane in the manner explained when describing the mode of projecting the cornice, (Pl. 3.) and the true lines of the impost, as seen standing out from the wall, will be obtained from these perspective profiles. From the nearness of the eye to the right hand wall, the projection of the impost will cut off a great part of the elliptic *spandrels* above it; it is important therefore to obtain the projection of these mouldings correctly*. The mitres of the plinth and of the impost, round the piers *h h* of the arch, are obtained, either by the methods explained in the account of the cornice just alluded to, or, if the scale of the drawing be small, they may be drawn by eye, reference being made to the plan for their apparent bearing with regard to the perpendicular lines.

The imaginary plane through D *d* will also cut the hand-rail of the balustrade at right angles to the straight part of it, and just where it begins to turn in a curve of double curvature round the circular end of the *well,* these two sections are easily got from a geometrical profile of the staircase, and the straight lines of the hand-rail are to be drawn to W and X, through the points of the profile of it on this plane. Another section of the hand-rail by a plane passing through D *d*, *parallel* to the side walls, must also be projected, and these will be sufficient for drawing the winding part on the first landing at the second; the horizontal straight part on the landing is to be drawn to Y, from this latter section.

The *niche and archway*, &c., in the left wall, correspond in width, height, and distance from the end walls, and must be projected accordingly: the arched heads are drawn in the same way as the great arch; the circular panels in the elliptic spandrels require no particular notice: they were introduced to illustrate, practically, the theorems of the varying inclination of the axes of elliptic projections of circles at different distances from the vertex. The projection of the arch of the niche inclines more than that of the archway, because this latter is farther from the intersecting line of their plane, though their centres are in the same perpendicular to that intersecting line; while the axis of the panel inclines on account of its being removed laterally from the vertex. The two doors and the window also do not require any description except that for the reason given for finding the apparent projection of the impost correctly, the entablature over the right-hand door must be designed with care, the profile of the mouldings by an auxiliary central plane, being obtained, for that purpose, in the same manner as those of the impost were.

---

* Architects, being aware of this effect of projecting mouldings in hiding part of the arch above them, always make the arch to spring from a line *above* the top of the mouldings; especially when such an arch and impost is much raised above the level of the eye.

Since we are accustomed to see objects lying on the level ground appear *above* one another, or approach nearer the horizontal line, in proportion as they are situated farther from the spectator (see pp. 177, 178), it has been asserted by some persons, that such a projection of a *descending* flight of steps as appears in the plate conveys no correct idea of the real inclination of the stairs, because the step really the lowest *appears above* the highest, or that nearest the eye, as it would do if the edges of the steps were lines on the level ground or really rising from the ground. This objection can only be made by persons totally unaccustomed to discriminate between the *apparent* and the real forms of objects, and therefore incapable of thoroughly understanding a perspective drawing.

The judgment is always exercised, though unconsciously perhaps, in conjunction with the eye, in deciding between the real and the visible forms of objects; and we compare any object which we do not know with others with which we are acquainted, in our endeavours to estimate the size and figure of the former. If we looked at the descending flight of stairs in nature through a tube, so as to exclude all other objects, we could hardly be deceived as to their real character; for any one who had ever seen a descending plane at all, would instantly perceive the aërial effect of distance on the stairs, as different from that which would modify any set of forms presenting the same outlines to the eye, but lying on the level ground. As far as the mere outline of such a descending flight of stairs projected on a plane is concerned, if this were isolated, the person looking at the lines could not tell what they represented; but when seen combined with other forms, as in the example in the plate, no one who ever stood on the landing-place of a staircase, and who ever thought of what he saw, could be deceived.

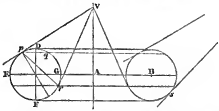
The horizontal lines of the impost, the top of the door, the bottom of the niche, and the lines of the horizontal *treads* of the stairs, all tend to one vanishing point, as we are accustomed to see them do in nature. Now the general outline of the flight of ascending steps, the skirting board, and handrail, have another vanishing point, and therefore the judgment acquired by experience in *looking*, informs us that that general outline and the other lines cannot be in planes parallel to those of the horizontal lines, but must be inclined to them. Again, the same lines, relating to the descending flight, are seen to have a third vanishing point, as different from that of the really horizontal lines as the one just alluded to belonging to the ascending stairs; and as from experience we know that a descending plane would bear such a relation of form to the horizontal plane, the eye cannot be misled simply by the circumstance of the lower step *appearing* above that in the foreground.

If the finite line A B (Fig. 1, Pl. V.) be supposed to turn round the point A, always moving in the same plane, a circle, having B for its centre, and its plane perpendicular to that in which the line moves, will form by its motion the solid of revolution called an *annulus* or *ring*.

This solid frequently occurs in architecture, as in the *torus* of the base of the column, &c.; it is necessary, therefore, that the draughtsman should be conversant with the mode of projecting it. The following properties, immediately deduced from the description of it, must be well understood for that purpose.

The line passing through A, perpendicular to the plane described by A B, is called the *axis*.

The centre B, and the two vertices of the diameter D F, of the generating circle, will, during the revolution, describe three



equal circles; the first lying in the plane in which the right line A B moves; and the other two each in a plane parallel to it, which two planes will be tangential to the solid in those circles. The two vertices E, G, of the diameter, of which A B is a prolongation, will describe circles concentric to that described by B, and lying in the same plane with it; and two concentric right cylinders, having their axes common with that of the solid, and A G A E, for the radii of their bases, will touch the annulus in these two concentric circles.

All other points in the circumference of the generating circle will also describe *parallel circles*, lying on the surface of the solid; the two produced by the extremities of any chord, as $p\,q$ parallel to A B, will be concentric, and will lie in a plane perpendicular to the axis.

If the solid be cut by a plane passing through the axis, the section will be two circles; or the generating circle in two positions.

If the solid be cut by a plane perpendicular to the axis, or parallel to that in which the radius A B moves, the section will be two concentric circles, the centre of which will be the intersection of the axis and plane.

If a tangent be drawn from any point, p, in the generating circle, not parallel to the axis, and meeting it in V; and from V another tangent, V r, be drawn to the generating circle; then V will be the common vertex of two concentric right cones, having their axes in common with that of the solid, and touching it in the circles produced by the revolution of the points p, r.

If the eye be at V in the axis, or if V be taken as the vertex, the circles described by p, r will be the apparent outline of the solid, and will be *perspectively* projected into circles, ellipses, or other conic sections.

But if the eye, or vertex, be not in the axis, and a plane be supposed to pass through both, the rays drawn tangents to the two circular sections made by this plane will touch them in four points, which will describe by their revolution four parallel circles, lying in four different parallel planes: and no two tangents drawn to the surface from the vertex, on the same side of this sectional plane, will touch the solid in the same parallel circle, nor can they touch it in either of the four parallel circles last mentioned.

The tangent drawn in the plane above-mentioned to the external surface of the solid nearest the vertex, will touch it in a parallel circle (s) lower down than that which is touched by the tangent drawn from the vertex on either side of the plane to the same part of the surface. And the tangent drawn in the sectional plane to touch the inner surface nearest the vertex, will touch it in a parallel circle *higher up*, or nearer the vertex, than any tangent not in that plane. If the first of these tangents be supposed to turn round always touching the solid, it will touch it in no plane curve, but in one of double curvature, the original of the visible external outline of the solid if projected; and if the second tangent be turned round in the same way, it will touch the solid, in another curve of double curvature, the *original* of the inner outline of the projection.

If one tangent from the vertex on each side of the sectional plane form equal angles with it, the two will touch the same parallel circle, provided they are drawn to the corresponding parts of the surface.

If the vertex be in the plane of any parallel circle, none of the interior surface can be seen; and of course less than half of the exterior surface will be seen.

To project the annulus, as seen from a vertex not in the axis, it is necessary to describe the conic sections which are the projections of as many of the *parallel circles* as may be necessary to furnish the projection of the curve which is the visible outline, for this curve cannot be projected into any of the usual *geometrical* plane curves*.

This is most easily, *practically*, done by projecting the two rectangular parallelopipeds, composed of six planes tangential to the solid, externally and internally; and having two faces in common, which are squares touching the annulus in two equal parallel circles, namely, those produced by the points D and F (see fig. p. 226); while the eight other sides touch it in the extremities of two diameters at right angles to each other, or in points corresponding to E and G of two vertical sections, by planes through the axis parallel to the sides of the parallelopipeds.

In Fig. 2, Pl. 5, an annulus is shown projected, C being the centre of the picture. To simplify the construction, the plane of projection is assumed parallel to the axis, and touching the edge H I of the exterior parallelopiped†, of which H I K L M N is the projection; H I being made equal to the diameter of the generating circle; and H O, N P, H Q, J R, being made *perspectively* equal to the same diameter (33, p. 145), the inner parallelopiped may be obtained by means of these points. At *a*, *d*, &c., are seen the elliptic projections of the circular sections of the annulus by two planes through its axis parallel to, and therefore bisecting, the sides of the rectangular solids; and at *m*, *h*, &c., are the projections of the sections by similar planes through the diagonals I L, K M, &c. of the parallel squares‡. The mode of obtaining these projections is familiar to the learner, and requires no comment, the principal lines of the constructions being shown in the figure.

It is obvious from the symmetry of the solids, and from the mode of generation of the annulus, that the eight points in which these circular sections touch the sides, or cut the diagonals of the circumscribing squares, will lie in the same eight parallel circles;

---

* Any curve lying wholly in a *cylindrical* surface may be *orthographically* projected (see *subs.*, p. 237) into the plane curve constituting the *base* of the cylinder; and any curve lying wholly in a *conical* surface may be *perspectively* projected into the curve of the base of the cone, if the vertex coincide with the apex of the solid, and provided, in both cases, the plane of projection be parallel to the base.

† The plane of projection is assumed oblique to the faces H K, H M, for reasons which will be explained in treating of the shadow of the solid; otherwise, the construction would be simpler by assuming the plane as coinciding with one of these faces, and therefore perpendicular to the other. (See p. 210.)

‡ The ellipses of three of these sections are not shown in the plate, to avoid confusing the subsequent constructions.

and the elliptic projections of these afford a guide, by which the curve representing the outline of the solid may be drawn by hand[*].

Besides the eight sections above mentioned, it will be advisable to draw those produced by a plane through the axis parallel to the plane of projection, when this is oblique to the sides of the rectangular solid, as in the present example; the projections of these sections will be circles (p. 196) as is shown at p and q. For these projections being easily obtained, and the circles described being perfectly correct, they will furnish additional and truer guides for the other ellipses, and for the outline curve.

The ellipses which would pass through the angles of the ten projected squares circumscribing these various sections, will obviously be the *projections* of the sections of the two right cylinders, touching the annulus internally and externally, by the upper and lower faces of the parallelopipeds.

If the vanishing point, W, of lines perpendicular to any plane, as (K N), cutting the solid, be found, then lines drawn from W, as W x, W y, tangents to the outer arcs of the elliptic projections of the sections of the solid made by that plane will be also tangents to the projection of the visible outline. For if a tangent be supposed drawn from the original point (X) of x, to the circular section made by the plane (K N); and another tangent be drawn in the plane of the parallel circle passing through X, and therefore perpendicular to the plane (K N); a plane passing through these two original tangents will touch the annular surface in X: and if X be so taken that this plane may also pass through the vertex, it will pass through the radial (Def. II. p. 128) of the horizontal tangent; W x, therefore, is the common projection of both original tangents, or of the whole plane touching the solid in X (2, p. 126), W x must therefore be a tangent to the outline, since all lines drawn through W represent lines perpendicular to the plane (K N); and the horizontal tangent is perpendicular to that plane, or to the radius of the parallel circle through X. The same reasoning will apply to W y.

If two tangents to the elliptic projection 1, 2, 1, 2, of the largest parallel circle, be drawn perspectively parallel to the projection of the axis, they will also be tangents to the outline of the solid: for these parallel tangents are the projections of the visible outlines of the right cylinder enclosing the annulus (see p. 206), which cylinder touches the solid in that parallel circle, and consequently these tangents must also be tangential to its outline.

If the axis, instead of being parallel, were oblique to the plane of projection, the principles of the construction would be the same; and the circumscribing parallelopiped must be projected as the cube was Fig. 1, Pl. II.

---

[*] It will be in vain to attempt to employ an elliptograph to describe the ellipses, for no accuracy, or facility of adjustment, will admit of their being drawn by means

§ ORTHOGRAPHIC PROJECTION.

If the distance of the vertex from the original object, instead of being at a *finite* distance, as it is always supposed to be in perspective projection, be assumed to be at an *infinite* distance; the rays from the object, instead of forming a *pyramid*, will form a *prism*, or will be parallel among one another; and the projection becomes what is called *orthographic*.

As radials or vanishing planes, supposed to pass through a vertex infinitely distant, can only meet the plane of projection in points or lines infinitely distant from the intersecting points of the original lines and planes, no vanishing points or lines of the lines and planes of an original object can be produced; or these vanishing points and lines may be also considered as at an infinite distance, and the projections of original parallel straight lines will be parallels, or may be conceived as converging to an infinitely distant vanishing point.

The parallel rays from an original object may cut the interposed plane of projection either perpendicularly or obliquely; but in either case it is obvious that all the rays must meet the plane at equal angles.

The projections of objects by parallel rays, perpendicular to the plane of projection, are called the *plans* or *elevations* of those objects, according as the plane is supposed to be *horizontal* or *vertical*. Suppose, for example, the object be a building, and the parallel rays horizontal, or parallel to the surface of water, to meet a vertical plane of projection at right angles, in this case an *elevation* of the edifice is produced. If the plane of projection, on the contrary, be imagined as horizontal, and the rays to be vertical, a *plan* of the building would be obtained. The *plans* and *elevations* made use of by architects, builders, and engineers, &c., are such projections drawn on paper, as if produced from imaginary models, as has been explained of the analogous perspective projections of similar objects. (See p. 180.)

If the parallel rays are not only perpendicular to the plane of projection, but also to the parallel planes of an object reducible to rectangular parallelopipeds, all the planes and lines perpendicular to the first set will be projected into straight lines and points, since the parallel projecting rays will entirely lie in, or coincide with, such planes and lines.

of that instrument, so as to pass through the correct points, especially if the centre of the picture be on one side of the axis of the solid. For in that case each ellipse will have its axes differently inclined; the artist *must*, therefore, trust to his eye and hand.

By a plan or elevation made under such conditions, the dimensions, or magnitude, of a rectangular solid which are in one direction, or in planes parallel to the plane of projection, can only be represented: to convey any idea, therefore, of the *whole* solid, two such projections, at least, on two planes perpendicular to each other are required, and these projections, being separate figures, cannot convey to the mind at one impression an idea of the form of the object.

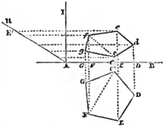
To obviate this objection, the parallel rays are, on many occasions, assumed as *oblique* to the planes of a rectangular solid, so that the surfaces of this lying in planes perpendicular to one another may be projected into figures, which will give an idea of the form and arrangement of the parts of the object such as a perspective projection would give; while the greater facility of practical construction of the orthographic projection causes it to be employed, though it cannot, from the supposed place of the vertex, give a *natural* representation, or such a one as will recall the idea of the original to a person not conversant with geometrical drawing, as the perspective projection would do.

It is obvious that equal parallel straight lines will be projected into straight lines, equal and parallel among themselves, but equal to, or less than the originals, according as these are parallel or inclined in a less or greater angle to the plane of projection; or greater than the original in some cases if the rays are oblique to the plane of projection. And it is also obvious from the parallelism of the rays, that equal segments of an original line will be projected into equal segments of the projection of that line.

This same conclusion may be arrived at by an extension of the theorems demonstrated p. 153; for if the *orthographic* be regarded as a *perspective* projection, the vanishing point being infinitely distant, the line harmonically divided being infinite, the two finite segments of it must be equal. (Geom. II. § 6.)

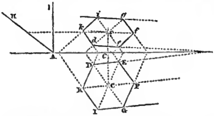
Though orthographic projection, theoretically considered, is only a modification of linear perspective, yet the practical geometrical constructions in its application are sufficiently different to require explanation and examples, which will now be entered on.

An original point or straight line may always be supposed to lie in a plane, the intersecting line A B of which, with the plane of projection, may be assumed at pleasure in that plane, as well as the angle of inclination of the two planes; for if the original plane be supposed first determined on, the plane of projection may be conceived to be taken, so as to form any convenient angle with the former: this angle must never be assumed as a right angle, for if it were, the point or line would be projected into one, coinciding with the intersecting line A B, the rays being perpendicular to the plane of projection, and therefore coinciding with the original plane. Hence the two planes are assumed as oblique to each other; and as then being brought to coincide in one plane by the revolution of either on A B, as is done in perspective projection. Let E be a point in the plane (A B)*, which is inclined to that of projection in the angle H A I. Draw the lines A I, A H, representing the two planes seen *edgeways* as it were, or representing their intersections with the drawing board considered as another plane of projection, and perpendicular to them both, A I being drawn perpendicular to A B. If A E′ in A H be made equal to *E* E, the perpendicular distance of E from A B, then lines drawn through E′ and E, parallel and perpendicular to A B, will give by their intersection the projection *e* of the point. For E′*e* will, obviously, represent the ray from the point, perpendicular to (A I) the plane of projection; and E *E* will be the *plan* of the same ray, as it were, on the original plane, or *E e* will be the intersection of a projecting plane passing through the point, and perpendicular to both planes or to A B. Consequently, *e* will be the point as projected by such a ray on the plane of projection.



In the same way the projections *e*, *d*, *f*, and *g* of the other points of any right-lined figure C D E F G may be found, and the projection of this may be thus obtained.

Let D E F G I K be given as a regular hexagon situated in an original plane (A B) as before: A I and A H being drawn, the



former perpendicular to A B from any point in it at pleasure, and A H to make with A I the proposed angle of inclination of the two planes.

Find the projection *c* of the centre of the polygon in the same way as *e* was found in the last example. Produce the diagonals and sides of the hexagon to their respective intersecting points in A B, the intersecting line of the original plane: then *d g*, *e i*, *f k* being drawn through *c* from the intersecting points of the diagonals, and *e f*, *k i*, *f g*, *d k*, *d e*, *i g* being drawn parallel to the former respectively from the intersecting points of the sides, these lines will give the projection *d e f g i k* of the figure.

---

* See 2d Note, p. 143.

Lines drawn perpendicular to A B, from the points of the original as in the first example, will verify or assist to find the projections of the original points.

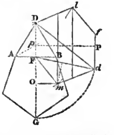
By this construction for the hexagon, the analogy between the orthographic and perspective projection is made obvious; the sides and diagonals of the projected figure *d e f k* may be conceived as drawn to infinitely distant vanishing points, in the vanishing line of the original plane. For as the vanishing plane is to pass through an infinitely distant vertex, it is obvious that it will cut the plane of projection (I A) in a vanishing line, infinitely distant from the intersecting line A B. And lines as C *c*, perpendicular to A B, may be regarded as drawn to the centre of that vanishing line, the originals, as C *c*, being perpendicular to A B. (See Def. 14, and pp. 149—156.) Or, to carry on the analogy, the lines as C *c*, K *k*, &c. may be conceived as the rays drawn to the infinitely distant vertex, this being supposed to be brought into the plane of projection, as was done in the example of the triangle. See Fig. p. 145.

It is obvious that if the precise angle of inclination of the two planes were immaterial, the point *c* might have been taken any where at pleasure in C *c* perpendicular to A B for the projection of the centre, provided *C c* did not exceed C *C*. It will be immediately seen, that the position of *c* would affect the resulting figure, which would naturally vary according as that angle of inclination varied. If *C c* were made equal to C *C*, it is obvious by the construction that the projection *d e f k* would be a regular hexagon equal to the original; this condition would imply that the original plane and plane of projection were inclined at no angle at all, or coincided, in which case of course the original and its projection would be identical. These two examples will sufficiently explain the mode of obtaining the orthographic projection of any right-lined figure.

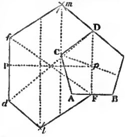
Let C D E (Pl. I. Fig. 3.) be given as the face of an icosahedron; A B the intersecting line of its plane, and V *a* P′ the angle of inclination of this to the plane of projection. Find by either of the foregoing methods the projection *c d e* of the given face; make *a* P″ equal to the perpendicular distance of P, the centre of the triangle, from A B; draw P′ *p*′ at right angles to A P″, and make it equal to the perpendicular distance between the opposite faces of the solid*. Draw $p'p''$ parallel to A B to cut P p in p'', then p'' is the projection of the centre of the triangular face parallel to C D E; through p' draw a line parallel to a P′, to cut V a in W; then a line drawn through W parallel to A B will be the intersecting line of the plane of this second face, opposite to C D E; and if the distance W p' of the centre of that opposite face from this intersecting line be set off from A B along P p, and the opposite triangle be drawn round the point as a centre, the projection $f\,g\,h$ of that face may be found in the same way as $c\,d\,e$. But as this would take up more room, a more convenient construction is adopted, which is shown in the figure.

Draw the face F G H as if projected on the plane of C D E, find the projection $f'g'h'$ of this triangle, and make $f'f$, $g'g$, $h'h$ in the perpendiculars to A B from f', g', h', equal to pp''; then $f\,g$ and h being joined, these lines will form the projection of the opposite face to $c\,d\,e$. The symmetry of the solid will admit of this construction being accurately verified in several ways, as will be readily understood from the figure without particular description.

On C D describe a regular pentagon (Plane Geom., Prob. 35,



* This distance is obtained by the following construction.
Let A B be the side of the solid; then an equilateral triangle, A B D, described on A B, will be a face. Draw D G perpendicular to A B, and find p, the centre of A B D: draw p P, perpendicular to D G, from p; make p P equal to twice p D, or the diameter of the circumscribing circle, added to the major segment of p D divided medially, (Pl. Geom., Prob. 6, Pr. 29.) p P will be the perpendicular distance between the parallel faces of the solid as required. Draw $d\,f$ through P, parallel to D F, and make P f, P d equal to p F, p D respectively, but in contrary directions: from D and d, with D F for a radius, describe arcs; and from f and F, with A B for a radius, intersect them in l and m; join D l, $f\,l$, F m and $d\,m$, the figure D $l\,f\,d\,m$ F D will be the section of the solid through the two parallel faces, by a plane perpendicular to them, and passing through D F. A regular pentagon on A B will be a section of the solid cutting off a pyramid of five adjoining faces (Geom. IV., Pr. 50, § 3). F d and F G will be equal.



The angle D F m is that contained by any two adjoining faces of the solid; the angle F D l is that formed by a side with the adjoining face; and D d will be the diameter of the circumscribing sphere.

If A B were the side of a dodecahedron, the regular pentagon on it, A B C D, being a face, and p the centre of this; then the perpendicular distance, p P, between two parallel faces, is equal to twice p D, or the diameter of the circumscribing circle, added to the major segment of p D divided medially; and the section of the solid D F $l\,d\,f\,m$ D is obtained by a construction similar to that just given for the icosahedron.

p. 61), and produce the sides of it G′ I′, G′ L′, to Q and R in C D produced: make F S, H T, in F H produced, equal to C Q, D R. Find the projections q and r of Q, R, in $c\,d$ produced, and the projections s, t, of S and T in $f\,h$ produced; draw $e\,s$, $e\,t$, $g\,q$, $g\,r$; draw $f\,o$ parallel to $e\,h$, and $h\,m$ parallel to $e\,f$. Also draw $d\,l$ parallel to $g\,e$, and $c\,i$ to $g\,d$; then $e\,o\,f\,h\,m$, and $g\,i\,c\,d\,l$, will be the projections of the two pentagons, which will lie in parallel planes, and cut off two opposite pyramids from the solid. (See Geom. IV. p. 158.) Then if $d\,k$, $g\,k$, be drawn parallel to $g\,m$, $d\,h$ to cut each other in k; if $f\,n$, $e\,n$, be drawn parallel to $e\,i$, $c\,f$, to cut each other in n; and if $c\,o$, $o\,n$, $n\,m$, $m\,d$, be joined, as well as $l\,h$, $h\,f$, $f\,i$, $i\,k$; and also $o\,i$, $c\,k$, $k\,d$, $f\,n$, $n\,h$, $m\,l$; the figure will be completed.

For C D L′ G′ I′ and H M′ E′ O′ F represent the originals of the two pentagons above referred to, brought into the planes of the faces C D E, F G H, respectively, by the revolution of the planes of the pentagons on C D, F H: now the points of Q, R, S, T, in which the sides of these figures intersect the planes of the two faces, will not be affected by the revolution of the planes just mentioned; the projections of the points Q, R, S, T, will therefore be the points through which the projections of the sides G′ I′, G′ L′, E′ O′, E′ M′ of the pentagons will pass, and are employed accordingly in finding the projections of the two polygons.

The intersections of these numerous lines will mutually verify each other—thus $c\,k$, $n\,h$ ought to be parallel, as ought also to be $f\,n$, $d\,k$, &c.; and it will be better to complete the projection of the solid on the plane C D E, as is shown at I K L M N O; which is readily done, as will be seen from the figure, by means of the points Q, R, S and T. Then perpendiculars to A B, from the points of this *plan* of the solid, will verify, or assist in determining, the corresponding points of the projection.

When the solid is composed of many faces, so that the edges are short in proportion to the bulk, the apparent convergence of the parallel edges is not so observable, and consequently an *orthographic projection*, in which these edges are necessarily drawn parallel, sufficiently resembles a *natural* or *perspective* view to allow of its being substituted for one, as may be seen from this example of the icosahedron compared with the perspective projection of the rhomboidal dodecahedron in Plate II.

Orthographic projection is hence particularly applicable to the representation of small bodies, as crystals, which would naturally be viewed from such a distance, in proportion to their size, that the rays would approach nearly to parallel lines, and on many occasions the construction of such projections is rendered considerably more easy by assuming the projecting lines as *oblique* to the plane of projection.

If one face of a cube be supposed parallel to the plane of projection, the projection of that face will be a square equal to the original, whether it be projected by lines vertical or oblique to the plane (47, p. 152); and if they are assumed as oblique, it is obvious that the adjoining faces may be projected into rhombuses, because such projecting lines will not in that case lie in the planes of those faces, as they would do if these lines were perpendicular to the plane of projection.

Draw the square G K J D (Fig. 5, Pl. II.) at pleasure; and through the four angles draw the equal parallels G F, K I, J H, and D E, to make any angle at pleasure with the sides of the square: join the extremities by F I H E, which will obviously form a square equal to the former, then this figure will be the correct projection of a cube by lines oblique to its faces, and therefore to the plane of projection, which is assumed as parallel to the opposite squares. For it is obvious that, however the lines through the angles be drawn, provided they are made equal and parallel, there must be some direction of the rays by which such a figure would be formed as the correct projection of a cube, having its face equal and parallel to the square D K.

Find the centres *l m n o p q*, of each of the six projected faces by drawing diagonals as I J, K H, &c.; draw the perpendiculars to each face *m* M, *n* N, &c., from these centres, parallel to the sides of the figure; thus *m* M, *p* P are to be drawn parallel to G K, D J; *o* O is to be drawn parallel to I K or D E, and so of the rest. Each of these perpendiculars is to be made equal to half the length of the projection of the side parallel to which it is drawn: thus *m* M, *p* P are to be made equal half G K; *o* O, *q* Q are each to be made equal half K I, and so of the rest: lines being then drawn from the extremities of these perpendiculars to each angle of the face on which it stands, the figure thus produced will be a projection of a rhomboidal dodecahedron (see p. 173).

By comparing this figure with Fig. 1, which is the perspective projection of the same solid, the learner will perceive the difference of form produced by these two modes of representing the same solid*.

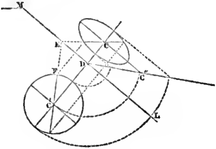
If the points *l m n o p q* (in Fig. 5) be joined, the projection of the octohedron included in the cube will be produced; and this will be the shortest mode of obtaining the orthographic projection of that solid, as it was formerly stated to be that for obtaining the perspective projection (see p. 173).

It is obvious that projection by lines oblique to the plane of projection can be applied to the representation of any parallelopiped, as well as the cube; but it is not necessary to give further examples, as the student who has made himself master of what has preceded, will find no difficulty in applying the same principles to any other figure.

---

* The perspective projection in Fig. 1 is an unfavourable example, the vertex being assumed too near the object; but this was obliged to be done, in order to bring as much of the construction as possible within the plate.

Any curve can be *orthographically* projected, as it can be *perspectively* projected, by finding the projections of a sufficient number of points lying in it, and drawing the projection of the curve through them by hand. But the *orthographic* projection of the circle, the curve of most common occurrence, is always either a circle or an ellipse, for the rays from the circumference form a *cylinder*, the section of which, whether it be right or oblique, is a circle or an ellipse. (Geom. App., Prop. 25, 26, and 27.) And the axes of the projection will be parallel and perpendicular to the intersecting line of the plane of the circle, if the cylinder of rays be cut by the plane of projection at right angles to its axis, that is, if the projecting lines or rays are perpendicular to that plane.



Let C′ be the centre of an original circle, E L being the intersecting line of its plane, and C D C″ the angle in which the plane of projection is inclined to the original plane, as before.

Now if an auxiliary plane be conceived to pass through the axis of the cylinder of rays, perpendicular both to the plane of projection and to the plane of the circular base, the intersections with these two latter planes will be perpendicular to the intersecting line E L of the plane of the circle, and will contain a greater angle than those of any other planes, equally passing through the axis of the cylinder of rays, but not perpendicular to both planes.

Thus if the auxiliary plane cut that of the original circle in D C′, and the plane of projection in D C, these lines will contain the angle L D C″, the planes being supposed in their relative positions. But if any other auxiliary plane cut the original plane in E C′, and the plane of projection in E C, the original angle of which D E C is the projection, formed by the two lines E C′, E C, will be less than L D C″.

Now all these planes will cut the circle in diameters, and the projection of that formed by the first auxiliary plane will be less than that of any other of the equal diameters formed by the other planes, this projection being as the cosine of the larger angle L D C″; consequently this projection being also a diameter of the ellipse and the shortest diameter, it must be the conjugate axis*.

Again, the diameter of the original circle which is parallel to the intersecting line E L of its plane, will have its projection equal to the original and also parallel to the same intersecting line, as is obvious from the figure, that original being parallel to the plane of projection; consequently this projection will be greater than that of any other diameter of the circle, none of which can also be parallel to the plane of projection, unless that plane be parallel to that of the circle, when there can be no intersecting line. This greatest diameter of the elliptic projection is, therefore, the transverse axis of the ellipse *.

If, therefore, the projection of the centre of an original circle be found, a line drawn through it, parallel to the intersecting line, will be the indefinite transverse axis of the elliptic projection; and this axis being made equal to the diameter of the original circle, the semi-conjugate is to be drawn equal to the cosine of the angle of inclination of the two planes, the radius of the circle being made radius.

If the original curve be an ellipse, or any other conic section, the projection of that curve will be a conic section; for if a cylindrical surface, as that formed by parallel rays from a conic section, be cut, any how, by a plane of projection, the section, when it is a curve, will be the curve of the base. Thus, for example, if the base be a parabola or an hyperbola, the cylindrical surface formed by parallel lines from it will always be cut in a parabola or an hyperbola; though these sections will only be *similar* to that of the base, when the plane of projection is parallel to that of the base.

---

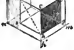
* If the cylinder of rays be cut *subcontrarily*, the circle will be projected into a circle, and the equal diameters will of course be represented by equal lines. (Geom. App. Prop. 26, and Cor.)

§ ISOMETRICAL PROJECTION.

It has been already remarked, that the principal lines and forms of most objects which are required to be represented by geometrical rules, are reducible to rectangular parallelopipeds having their faces respectively parallel. When such objects are to be delineated by orthographic, instead of perspective projection, this is usually done by means of plans, elevations, and profiles, or projections on three planes assumed parallel to three faces of the solid, as has been explained (p. 230).

Professor Farish, of Cambridge, has suggested an application of the principles of orthographic projection, by which such solids can be represented with their three pair of planes in one figure, which gives a more intelligible idea of their form than can be done by a separate plan and elevation. At the same time, the method admits of their dimensions being measured by a scale as directly as by this latter mode of delineation. The inventor termed this Isometrical Perspective, but, in accordance with the definitions already given, it will be called a projection in this work.

The projection of a right line, by projecting lines perpendicular to the plane of projection, is as the cosine of the angle of inclination of the original line to the plane; consequently, if finite lines are inclined in equal angles to that plane, their projections will be directly proportional to the originals.

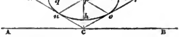


If the three plane right angles of a trihedral right angle are equally inclined to the plane of projection, and equal distances be set off along the edges from the angular point, the three lines joining the extremities will obviously form an equilateral triangle, the plane of which being parallel to the plane of projection, it will be projected into an equal equilateral triangle, as dhf: the centre C of which will be the projection of the angular point of the trihedral; and the equal angles $d\,C\,f$, $h\,C\,f$, $h\,C\,d$, will be the projections of the three plane right angles of it, and must each be equal to 120°. Through d, h, and f, draw lines parallel to $C\,d$, $C\,h$, $C\,f$; these will obviously be the projections of lines parallel to these last, and will meet in the points g, h, i; the figure $d\,e\,f\,g\,h\,i$ will form a regular hexagon from the construction; and as the lines $C\,d$, $C\,f$, $d\,e$, $d\,f$, are all equal, and are the projections of lines lying in one plane, and since the angle $d\,C\,f$ has been shown to be the projection of a right angle, the rhombus $C\,d\,e\,f$ is the projection of a square, and for the same reason so are the rhombuses C*fgh*, C*hid*, which are equal to the former; therefore, the whole figure is the projection of a cube, having all its faces equally inclined to the plane of projection: this is hence termed an isometrical[*] projection of that solid.

A right angle in this case is projected into either one of 120°, as *d* C *e*, *dih*, or into one of 60°, as C *di*, C *de*.

If points, as *d'*, *h'*, *f'*, be taken any where at pleasure in C *d*, C *h*, C *f*, produced, and lines be drawn through them parallel to the same lines respectively, the figure *d' lf' mh' n* will obviously be the isometrical projection of a rectangular parallelopiped, the sides of which are directly proportional to C *d'*, C *h'*, and C *f'*; its plane right angles being projected as before into angles of 120 and 60°. Hence if the three dimensions of the original solid be taken from any scale of equal parts, and laid along the radii from C, the figure of its projection may be drawn, and any proportional parts of those original dimensions will be represented by proportional parts from the same scale, set off either along the same lines, or along any others parallel to them, as is clear from the principles of orthographic projection.

But it must be distinctly understood that it is only on the three radii C *d'*, C *e'*, C *f*, or on lines parallel to them, that magnitudes can be set off or taken from the scale, so as to represent the projections of proportional magnitudes on the original: thus, for example, the projected diagonal C *l* will not bear the same proportion to its original, as the sides C *d'*, C *e'* do to theirs, because the original of that diagonal is not inclined in the same angle to the plane of projection; but it is seldom necessary to measure or set off distances on such lines, in the delineation of the class of objects to which this species of projection is applied; and when it is necessary, a very simple construction will allow of its being done[†].



It is obvious that all circles lying in planes equally inclined to the plane of projection will be projected into *similar* ellipses, because the circumscribing squares will be projected into similar rhombuses. Let C *def* be the isometrical projection of a square; then, from the principles of projection, *lmno*, the central points of the four sides will be the points at which the ellipse representing the inscribed circle

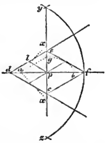
---

[*] The term is from two Greek words meaning 'equal measures.'

[†] If a finite straight line be projected on a plane by parallel rays, perpendicular to that plane, the length of the original may be ascertained from the projection, if the angle of inclination of the line to the plane of projection be known.

Let P Q be the orthographic projection of an unknown original finite line inclined will touch those sides: the diagonal df is, from the principles of this isometrical projection, parallel to the plane of projection, and will therefore be equal to the original, or to the diagonal of the square; and ab, the major axis of the ellipse, will be in this diagonal, and will be equal to the real size of the side of the square. If lq be drawn parallel to C e, or perpendicular to df, it will bisect pd in q, and from the property of the ellipse pa is a mean proportional between pq and pd; or ab is a mean proportional between pd and df: gh, the minor axis, is a mean proportional between pe and C e; and gh, the minor axis, will be to ab the major, as C e to df: ol, mn, are termed isometrical diameters, and are obviously conjugate to each other.

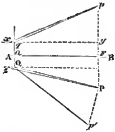
From these properties the following simple constructions are deduced, by which the various points required in an isometrical projection are easily found. Let ab be given as the side of an original square, bisect it in p by a perpendicular to it; make px equal to pa or pb: then set off the length of the diagonal ax, each way from p, along ab produced to d and f: from d or f, with df for a radius, describe a segment of a circle, and from f or d, with the same radius, intersect it in y and z; bisect the arcs fy, fz, by lines from d which will cut px in e and c: then fe, fc being drawn, they will complete the projection of the square. Lines through a and b parallel to the sides, will give the extremities



at a given angle to the plane. Make P Q p' equal the given angle, and draw P p' perpendicular to P Q, then p' Q is the length of the original sought.

For P Q p' is the projecting plane of the line turned down on P Q, its intersection with the plane of projection, and thus brought to coincide with it: p' Q therefore represents the original line.

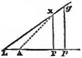
If the angle at which the line was inclined to the plane of projection were not known or given, another datum is necessary to find the length of the original line: this other datum is generally the projection of the same line on another plane perpendicular to the former, and intersecting it in any given line A B. Let pq be the projection of the same original on this second plane by lines perpendicular to it, and therefore parallel to the first plane.



Draw qy parallel, and py perpendicular to A B; make yx in yq equal to P Q: then px will be the length of the true sought, and yxp will be its angle of inclination to the first plane of projection.

For if the first projecting plane of the line be supposed to turn round on the line py, which represents on the second plane the projecting line of the point P; till it be brought parallel to this second plane of projection, then the triangle formed by the line, its projection P Q, and the projecting line of the point P, will be represented on of the minor axis of the inscribed ellipse, and ab will be the major.

If A P, P X be drawn at right angles, and equal to each other; and P L in P A produced, be made equal to A X, the diagonal of the square: then a line being drawn through L and X, the figure will furnish a scale for finding the axes, and the isometrical diameters of the isometrical projection of a circle.



For if L p in L P be made equal to the isometrical diameter, which is always the datum of each projection; and pg be drawn perpendicular to L p to meet L X in g: then L g will be the major, and pg the minor axis sought. For (see preceding figures) $pg : pa :: pe\ (pl) : pd$: and if pe be assumed as unity, de, or $2\,el = 2\,pe$, will be equal to 2; and $pd^2 = de^2 - pe^2 = 4 - 1 = 3$, or $pd = \sqrt{3}$. Therefore, by similar triangles, if pg be assumed as 1 or $\sqrt{1}$, pa will be $\sqrt{3}$: and since by the property of the ellipse $2\,pl^2 = pa^2 + pg^2$; $2\,pl^2 = 3 + 1 = 4$ or $pl^2 = 2$ and $pl = \sqrt{2}$*.

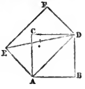
Now (see figure above) if P X be assumed as $\sqrt{1}$, AX^2 (LP^2) $=$ 2, or A X (L P) $= \sqrt{2}$ and $LX^2 = 2 + 1 = 3$, or $LX = \sqrt{3}$. Consequently, P X, P L, L X, or the sides of any triangle p L g, similar to P L X, will represent the minor axis, the isometrical diameter, and the major axis of the ellipse.

---

the second plane of projection by one equal to it. $pxy : px$, therefore, will be equal to the line sought, and pxy will be its angle of inclination to the first plane of projection.

This construction is employed by carpenters to ascertain the length of the *hip* of a roof, of which they have only the plan and elevation. For this hip, being *oblique* to the two planes, on which the building is supposed to be projected, is of course projected into a line unequal to the original. It is obvious that P Q, or its equal xy, is the cosine of the angle of inclination of the line to the plane as was stated above in the text.

* Hence the minor axis, the isometrical diameter, and the major axis of a circle isometrically projected, are in the same ratio as the side A B, the diagonal of a face, A D, and the diagonal of the solid, DE, of a cube. For $AB = \sqrt{1}$, $AD = \sqrt{2}$, and $DE = \sqrt{3}$, as is obvious from the figure: A D E F being a section through the cube by a plane through the parallel diagonals of two parallel faces, A E being made equal to A B.



§ PROJECTIONS OF THE SPHERE AND OF ITS CIRCLES.

The principles of orthographic and perspective projection are applied to the delineation on a plane of the circles of the sphere, as representing the earth, or the imaginary sphere to which the stars may be referred, in the construction of geographical and astronomical maps.

The situation of the *vertex* gives rise to the various modes of representing the spherical surface, which have received different names accordingly. These names will be retained for the purpose of distinction, though in fact all the varieties of projections of the sphere are reducible to the two classes; *perspective* projection, the vertex being at a definite distance; or *orthographic* projection, the vertex being supposed at an infinite distance.

It is obvious that equal portions of the surface of a sphere will be projected on a plane into very different figures of unequal areas. If a line be drawn from the vertex to the centre of the sphere, the part of the surface surrounding the point where the line cuts it, will be represented in the projection more nearly of the same form as the original, than the portions lying nearer to the visible boundary; provided the plane of projection be perpendicular to the line from the vertex to the centre.

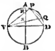
In the construction of maps, especially geographical, in which it is desirable to have the various countries represented in their correct proportions and figures, that position of the vertex should be taken, which will allow of this condition being fulfilled as nearly as possible.

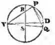
But it is also necessary that the curves representing the circles of the sphere should be such as may be drawn with tolerable facility; hence the former condition is sacrificed to this, and the different kinds of projection above alluded to have been employed, according as the one or other of these requisites were considered most important.

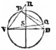
If the vertex be taken in the surface of the sphere itself, and the farthest or concave hemisphere be projected on a plane passing through the centre of the sphere perpendicular to a line from the vertex to that centre, the projection is commonly called *stereographic* *, and is the one most usually employed, because *all* circles on the spherical surface are in this case projected into *circles*, and not into *ellipses* or the other conic sections.

---

* This word, derived from the Greek, means "delineation of a solid," and is therefore just as applicable to all kinds of projection of any solid, as to the one in question of a sphere.







Every circle on the surface of a sphere may be considered as produced from the section of the solid by a plane, in which the circle and its centre will of course lie. Another plane may always be conceived to pass through the centre of the sphere, the vertex, and through the centre of the circle; the section of the solid by this second plane will be what is termed a *great circle* of the sphere.

Let V A B D be such a great circle, V being the place of the vertex, and A B, at right angles to V D, being the *intersecting line* of the plane of the circle V A B D with the plane of projection; these two planes being perpendicular to each other, and the former being also perpendicular to the plane of the original circle. Let P Q be the intersection of the plane of this original circle, with the plane of the great circle V A B D, then P Q will be a diameter of the original circle; draw V P, V Q, and P D.

The angles V D P and V Q P are equal (Geom. III., Prop. 15), as are also the angles D V Q, D P Q; and the angles V P D, V C R are right angles:* then the angle V S R = D V Q ± V C R, and V P Q = V P D ± D P Q, therefore V S R = V P Q; and the angle P V Q being common to both, the triangles P V Q, S V R are similar; therefore the section of the cone of rays from the original circle, by the plane of projection, is subcontrary. (Geom. App. Pr. 23.)

Hence in a stereographic projection of a sphere, if the projections of three points in the circumference of an original circle are obtained, a circle described to pass through them will be the *projection* of that original circle.

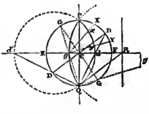
The plane of projection will cut the sphere in a great circle; this may be called, for distinction, the *primitive*, and may be drawn at pleasure according to the assumed magnitude of the sphere. The centre C will be the centre of the picture, and the distance of the vertex is equal to the radius of the sphere; therefore, if this radius be turned down in any direction on the plane of projection, it will always bring the vertex into the circumference of the *primitive*. The plane of any other great circle† will cut the plane of projection in a diameter of the primitive, and this diameter P Q may be drawn on the paper at pleasure.

Let this circle be inclined to the plane of projection in any angle, for example 37°; draw E F perpendicular to P Q, make

---

* C is the centre of the circle V A D B; but is not marked in the figure.

† That is, any circle the plane of which passes through the centre of the sphere.



E C D equal to 37°, and draw Q D, Q D to cut E F in d and d'; through P, Q, d, and d', describe a circle, which will be the projection of the original one forming the given angle with the plane of projection.

For E F will be the intersecting line of a plane, perpendicular both to the plane of projection, and to the original circle, and will therefore pass through the vertex; if this plane be turned round on E F till it coincide with the plane of projection, the vertex will be brought to Q, and D D will be the intersection of the original plane with this auxiliary plane; Q D will hence represent the ray from the point in which the auxiliary plane cuts the original circle; d, d' will, therefore, be the projections of D, D, and will, consequently, be points in the circle which is the projection of the original one.

To draw the projection of another great circle, having P Q *also for its intersecting line, and making any proposed angle with the former circle* P d Q.

Draw the diameter G G, to make the proposed angle with D D; draw Q G to cut E F in g, g, then a circle through P, Q and g will be the projection required.

This is evident from the mode in which the former circle, P d Q, was drawn to make a given angle with the plane of projection.

To find the projections of two points in any original great circle (P d Q) *which shall comprise an arc of a given number of degrees.*

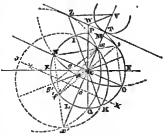
The projection x of one point being assumed or given, draw E F, as before, at right angles to P Q, to cut the given circle in d, d'; draw Q d produced, to cut the primitive in D; draw C D, and make D C G a right angle, or make the arc D G a quadrant of the primitive; draw Q G to cut E F in g, g', draw $g\,x$ to cut the primitive in X; set off the arc X Y, on the primitive, of the proposed number of degrees; draw g Y to cut the projected circle in y, then y will be the other point sought; or $x\,y$ will be the projection of the proposed arc.

For if Q R be drawn parallel to C D, cutting E F in R, then a line through R parallel to the intersecting line P Q, will be the vanishing line of the plane of the circle (P d Q), and Q R is the principal radial. (Def. 14, p. 132.) Now the angles E C G, P C D, C Q R, are obviously equal, therefore g C G, (E C G)

$+$ CGg (CQg), are equal to CgQ (RgQ); but CQR $+$ CQg are equal to RQg; consequently RQg, RgQ are equal, or $RQ = Rg$*.

Hence g represents the vertex, brought into the plane of projection by the principal radial being brought into it, and therefore the primitive representing the original circle also brought into the plane of projection, rays from g through X and Y will cut the projection of that original circle in the images of those points; or the arc X Y will be the original of $x\,y$. (See p. 145.)

The above construction for finding the point g was given, instead of making Rg at once equal to R Q, the principal radial, partly because it is practically more accurate, and partly because the point g is that through which the projection of a great circle will pass, which is perpendicular to the original one. The original point (G) of g being 90° distant from the circumference of the original circle, g is called the *pole* of the circle $P\,d\,Q$.



As every point x, in the circumference of a great circle, must be in the spherical surface; x is one extremity of a diameter of the sphere of which $M\,x\,C\,L$ is the projection; if N O be drawn at right angles to M L, and a line be drawn from N through x, and $N\,x'$ be drawn perpendicular to $N\,x$, $N\,x'$ will cut M L produced in x', the projection of the other extremity of the diameter; and x' will be in the circumference of the circle $P\,x\,Q$: this is obvious from the principles of the projection.

Hence all great circles which pass through the original of the point x must also pass through that of x', and therefore the centres of the *projections* of such great circles will lie in a line J K, bisecting $x\,x'$ at right angles.

To draw the projection of a great circle which shall pass through the extremities of a given diameter ($x\,x'$),† *in the plane of a given great circle* ($P\,x\,Q\,x'$), *and shall be inclined to this circle in a given angle.*

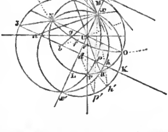
Draw a diameter to the primitive, N O, perpendicular to $x\,x'$; draw lines from N, or O, through x and x', to cut the primitive in s, f'; through either of these points, as s, draw a tangent to the primitive (Plane Geom., Prob. 40, p. 70), cutting $x x'$ produced in T; through T draw a line Z T, parallel to N O, and produce Q P, the intersecting line of the given circle, to cut Z T in W; join W x,* make the angle W x Z equal to that which the planes of the two great circles are to form; then a diameter Z H, to the primitive, drawn through Z, will be the intersecting line of the second circle required, the centre of its projection will be obtained by a perpendicular bisecting I H at right angles, cutting J K in the centre sought.

---

* R, the centre of the vanishing line, is therefore the centre of the projected circle, ($P\,g\,Q$) perpendicular to the original circle ($P\,d\,Q$).

† When the letters are inclosed in brackets, it indicates the originals of which the line or circles so marked are the *projections*.

Make T V, in $x x'$ produced, equal to s T, join W V, Z V. It is obvious that Z T is the intersecting line of a plane, tangential to the sphere at the *original* of the point x; and that V represents that original point brought into the plane of projection by the revolution of this tangent plane on Z T; hence W V, Z V are the intersections of the planes of the two great circles with this tangent plane, W x, Z x being the projections of these intersections; and since this tangent plane is perpendicular to a diameter of the sphere ($x x'$) drawn from the tangent point, the plane is perpendicular to all planes passing through that diameter, as those of the two great circles do; consequently, (Geom. IV., Prop. 17, Schol.) the angle Z V W is equal to that formed by these two planes. But the angles N O s, N s T are equal. (Geom. III., Prop. 17.) N s O, N C x are both right angles, and O N s is common to the two triangles, therefore N x C, or T x s, is equal to N O s, or to x s T; consequently T x is equal to T s or to T V; and the angle Z V W, which is that formed by the planes of the great circles, is equal to the angle Z x W, which the projections form with each other; for Z x, W x are tangents to those projections, being the *projections* of tangents to the original circles. (Geom. App. Prop. 10.)



From d, the point in which J K cuts $x x'$, describe a circle to pass through N, O; this circle will also pass through x, x' on the principle of the preceding construction and demonstration; and this circle will be the projection of a great circle of which N O is the intersecting line, and having a common diameter ($x x'$) with the two former. Find the poles e, e'† (see

---

* Neither W x nor Z x are drawn in this, but are shown in the figure in this page; which must be referred to for those points and lines not seen in the former one.

† e' is not in the figure, but its place in x' T will easily be understood.

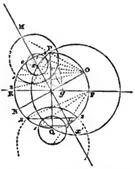
p. 246) of $N x O x'$, and describe a circle $N e O e'$, the projection of an original great circle, passing through them; the plane of this circle will, therefore, be perpendicular to that of $N x O x'$, or perpendicular to $(x x')$; it will, consequently, be also perpendicular to the planes of all great circles (as $P x P$, $I x H$), which have $(x x')$ for a common diameter: the circle $N e$, $O e'$ will therefore pass through the poles of all such great circles (see both figures), the points $f g$, in which it cuts diameters of the primitive perpendicular to the intersecting lines $I H$, $P Q$, will be the poles of the two great circles ($I x H$, $P x P$), (see p. 246) and as (x, x') are, conversely, the poles of the circle ($N e O e'$), lines drawn from x, or x', through $f g$, will intercept an arc of the primitive which is the measure of the angle formed by the planes of those two great circles.

It has been proved above, that tangents at x or x' to the projections of all great circles, having $(x x')$ for their common diameter, form the same angles with each other that the planes of the original circles themselves form; and if radii be drawn from $x x'$ to the centres of these projections, in $J K$, they will, of course, form the same angles as the tangents, that is, $a x b$ is equal the angle $W x Z$.

Draw lines from x or x' through h and p, the points in which the projected circles $I x h$, $P x p$, cut $J K$, to cut the circle $N x O x'$, in h' and p'; and join $d h'$, $d p'$; then since the arc $x p x'$ is double the arc $p x'$, an angle standing on that arc is double the angle $x' x p$ standing on $p x'$: therefore the first angle is equal to $x' d p'$; but that first angle is also equal to $T x W$ (Geom. III., Prop. 17). In the same way, the angle $x' d h'$ may be proved equal to $T x Z$; therefore $p' d h'$, the difference of the two $x' d h'$, $x' d p'$, is equal to the angle $W x Z$, or to the angle formed by the planes of the original great circles. Hence is derived the following construction.

If the arc $p' h'$ on the circle $N x O x'$ be made of the same number of degrees as the proposed angle of the planes of the two great circles, and $x p'$, $x h'$ be drawn, they will cut the line $J K$ in points through which the *projections* of the great circle will pass.

If, therefore, a circle be described on $x x'$, the projection of any diameter of the sphere, and a diameter, $J K$, be drawn at right angles to $x x'$; then by dividing the quadrants into equal arcs, of any proposed number of degrees, and by drawing lines from x to the points of division, these will cut $J K$ in points, through which the projections of great circles will pass, which form those proposed angles with each other and have $x x'$ for the projection of their common diameter.



Small circles of the sphere, or those of which the planes do not pass through its centre, require more data in order to find their projections than are necessary for a great circle; two of these data are the *poles* of the circle, or the points in which a diameter of the sphere, perpendicular to the plane of the circle, and therefore passing through its centre, cuts the spherical surface; another datum is the number of degrees of the arc of a great circle passing through these poles; which the small circle intercepts from either of them.

Let $x\,x'$ in the great circle $P\,x\,Q\,x'$ be the poles of a series of parallel small circles at 20°, 40°, &c., from x. Draw a diameter, L M, to the primitive, through x, x', and another N O, as usual, at right angles to it. Draw O x, to cut the primitive in o; from o set off, each way, on the circumference of the primitive, 20°, 40°, &c., to 1, 2, &c.; then lines drawn to O from 1, 2, &c., will cut L M in points, through which circles are to be described as the projections of the proposed small circles. If g, the pole of $P\,x$, $Q\,x'$ be found, and a line be drawn from g through x, to cut the primitive in a point, and 20°, 40°, be set off on each side from it; then lines drawn to g from these points will cut the original circle the points through which the small circles will pass (see p. 245), as is seen by the figure.

If the poles of the small circles were in the primitive, as, for example, at P and Q, the arcs 20, 40, &c., must be set off on each side, as from Q to 1, 2, &c., then lines must be drawn from F to these points; which will cut the diameter P Q in the extremities of the diameters of the projections; and the circles being described to pass through them, they will also pass through 1, 2, &c., in the primitive. For the planes of the original circles are in this case cut by the plane of projection in diameters, namely in the chords 1 1, 2 2, &c.

These constructions are immediately deduced from the principles already explained. By them, all the geographical and astronomical circles of the sphere in any position may be described, on the principles of this kind of projection. There are three of these positions usually in use.

1. When the axis of the earth, or celestial sphere, is in the plane of projection. The meridians are drawn in the manner explained, p. 245, to form the proper angles with each other; and the parallels of latitude are drawn, as explained above. This does not require any further illustration.

2. Fig. 1, Pl. 8, is the stereographic projection of the sphere, the axis being oblique to the plane of projection, and lying in a plane passing through the vertex, and therefore perpendicular to that plane.

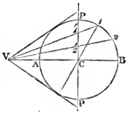
The arc Z P is made equal to 51½°, the latitude of London, consequently the centre of the primitive would represent that place in a map drawn on the figure. The meridians are obtained by the constructions, pp. 246, 247, and the parallels of latitude by that explained p. 249.

When the stereographic projection is employed for maps of the world, the countries which fall near the primitive are represented much too large in proportion, as will be perceived from examining the figure just cited, where, as will be seen, the equal arcs of the meridians and parallels of latitude are projected into larger arcs, near the primitive than near the centre. It must be observed that in employing this projection for this purpose, the outline of the land and water is not *reversed;* as it ought strictly to be, from its being the *concave* farther hemisphere that is looked at; but the right hand side of the projection is taken as the eastern, and the left as the western side of the globe, as if it were the *external* hemisphere which was represented. Of course there must be two projected hemispheres, in order to show the whole surface of the globe.

3. If the vertex be at either pole, the meridians will be projected into radii of the primitive, which will in this case represent the equator; the parallels of latitude will be projected into concentric circles, and the diameters of these are obtained by drawing lines from the extremity of *any* diameter of the primitive to cut another drawn at right angles to it, from the equal arcs of the primitive (p. 249).

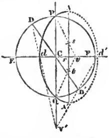
If the vertex be any where either within, or without, the spherical surface, the circles of the sphere will be projected into ellipses or other conic sections. And as not only the place of the vertex, but that of the plane of projection may be varied, there may be various kinds of projections; but in all, it is obvious that the constructions must be derived from the principles of perspective projection.

If A P B be a great circle of the sphere, passing through the vertex, and P P be the plane of projection passing through the centre of the sphere, perpendicular to the plane of the circle, then V the vertex being taken, so that V A is ·70 of the radius A C, equal arcs of P B, as P 1, 2, &c., will be projected into



nearly equal segments P 1′, 1′ 2′, &c., of the diameter*. Hence, for the reasons before given, this is the most accurate projection for the construction of maps, and has been called the *globular projection.*

The principles of the following constructions for drawing the circles of a sphere on this projection are easily deducible from the rules of perspective, and from what has been already given of stereographic projection.



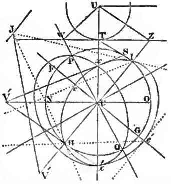
Let P Q be assumed as the intersecting line of the plane of an original great circle inclined to the plane of projection in any angle, as 55°. Draw E F perpendicular to P Q, and make the angle F C D, 55°; draw the diameter D D′. Make C V′ in C Q produced equal 1 · 68 of the radius C Q, then draw V′ D, V′ D′ to cut E F produced in *d d′*. Bisect *d d′* in *r*, and draw a line through *r*, parallel to P Q. From P and Q with *r d* or *r′ d* for a radius intersect the perpendicular through *r* in *s* and *t*; draw P *s*, Q *t*, which will of course intersect in a point *v*, in *d d′*: then *v* P or *v* Qbeing set off each way, along the perpendicular from *r* to A, A′; A A′ will be the one, and *d d′* the other axis of the elliptic projection of the proposed circle.

This construction for finding the other axis of an ellipse, one axis, *d d′*, and a point in the curve being given, is that alluded to in note J J, explaining the principle of the trammel; but the other axis might be obtained by the principles of perspective projection; for a line drawn through V′ parallel to D D′ will cut E F, or that line produced, in the centre of the vanishing line of the plane of the original circle: if this vanishing line be drawn accordingly, parallel to the intersecting line P Q, and the length of the principal radial be set off each way from the centre along it, the projection of the circumscribing square may be obtained (see p. 149), from which the axis required may be deduced by Prob. 60 of Plane Geom. (p. 90.)

---

* If the diameter P P, and the circumference P B P, be divided into the same number of equal parts, the lines drawn through the points of division in succession will not cut B V in any one point; so that there is none, from which, as a vertex, equal portions of the spherical surface will be projected into equal areas. The decimal · 70 in the text is the mean, obtained by calculation, of the distances from the point A, in A V, at which the lines drawn from equal divisions of 10° through equal segments of P P, cut this line A V; · 75, or three-fourths of the radius, is also employed for the same distance; as being more readily laid off on the drawing.

To find the intersecting line of another great circle inclined to the first in a proposed angle, and having xx' for the projection of their common diameter.



Draw a diameter, N O, to the primitive, perpendicular to xx', and make C V' in N O equal to the distance of the vertex; draw V x to cut the primitive in S; from S draw S T a tangent to the primitive, cutting xx produced in T; and draw W T Z parallel to N O. Make T U equal to T S; produce Q P to cut Z T in W; join W U, and make the angle W U Z equal to the proposed inclination of the planes of the two circles; then a diameter, I H, to the primitive, drawn through Z, will be the *intersecting line* of the second circle.

Draw the diameter F G at right angles to I H, and make C V' in I H, produced, equal to the distance of the vertex: make the angle I V' J equal to the angle T U Z, or equal to the complement of the inclination of the second grent circle to the plane of projection, V J cutting F G produced, in J; through J, parallel to H I, draw the vanishing line of the plane of the circle, and make J K* in the vanishing line equal to J V' the principal radial; then lines drawn from K through I and H will cut F G in e' e, the vertices of the one axis, and those of the other must be found as explained above, from one axis ee', and a double ordinate H I, or by completing the *perspective* circumscribing square, and by Prob. 60, Plane Geom. (See p. 90.)

It is obvious that the latter part of this construction, by which the projection of the vertices of one axis is found by means of the vanishing point of the diagonals of a square circumscribing the original circle, is equally applicable in the former construction, in order to obtain the other, when the one axis was obtained.

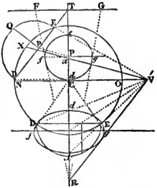
The rays from the vertex tangential to the spherical surface form a right cone, which touches the sphere in a small circle parallel to the plane of projection; and since all the great circles *touch* this circle, their projections must *touch* that of this small circle, that is, the ellipses, as found by the preceding construction, will touch a circle concentric with the primitive, and which represents the outline of the sphere, as seen from the given point of view; lines drawn from V, tangents to the primitive, will cut xx

---

* K is not within the figure.

produced in the diameter of the circle, which is the projection of this visible outline of the sphere.

Although, for the sake of making the principle clearer, the circle from U was directed to be described with the radius U T; yet it is obvious that the angle W U Z might be set off on the circumference of any circle described from U as a centre: it will be, therefore, better in practice to describe a larger circle, at pleasure, from U, for this purpose, the radii of which will of course cut the intersecting line Z T in the same points Z, W. Or, since T W, T Z are the tangents of the angles T U W, T U Z, to the radius T U, those tangents may be taken from a table of natural tangents, and set off on the line, from a scale of which T U is the unit.



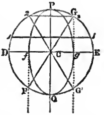
Let $x\,x'$ be the projection of a diameter of the sphere; draw N O at right angles, C V′ being made equal to the distance of the vertex; draw V′ x to cut the primitive in X; from X set off, each way, on the circumference any proposed arc to D and E, and draw the chord D E; draw lines from D and E to V′ to cut $x\,x'$ in d, e; bisect $d\,e$ in P, draw V′ P to cut D E in p, draw P Q perpendicular to D E, to cut a semicircle described on D E in Q. Produce D E to cut $x\,x$ produced, in T, and draw the intersecting line of the plane of the circle through T perpendicular to $x\,x$; draw the indefinite axis of the elliptic projection through P, parallel to this intersecting line. Through V′ draw the radial V′ R parallel to D E, to cut $x\,x'$ in R; set off the length of the ordinate P Q from T each way, along the intersecting line to F and G; then lines drawn from R to these points will cut the indefinite axis of the elliptic projection of the small circle in f, g, its vertices.

Many other equally obvious constructions may also be employed for the same purpose, derived from the principles of perspective; but this is the easiest and most accurate mode of finding the axes of the elliptic projection of a small circle of the sphere. And the same method is applicable, as will be seen from the figure, when the axis of the small circle is in the plane of projection: in this case the elliptic projection will, of course, pass through the extremities of the chord D E; this line being also the intersecting line of the plane of the circle.

When the axis of the globe is assumed as in the plane of projection, the mode of projecting the circles of the sphere is analogous to that of the stereographic projection, under similar conditions, and being explained in the preceding constructions, it requires no specific explanation; but from the difficulty of describing ellipses, an approximative figure, resembling the true globular projection, is commonly employed in this case, in which the circumference of the primitive and the two diameters at right angles to each other are divided into equal parts; and segments of *circles* drawn through the points, instead of segments of *ellipses*.

The sphere with its axis oblique to the plane of projection is seldom represented on the globular projection, from the trouble of finding the *axes* of the meridians, &c. in addition to that of drawing the curves themselves. In the atlas to the Encyclopædia Metropolitana, however, a splendid example of the two hemispheres, on this oblique projection, may be seen; and by comparing the forms and magnitudes of the different countries as there represented, with the same hemispheres projected stereographically in common maps, the superior merits of the globular projection will be obvious.

If the vertex be supposed at an infinite distance, so that the projecting rays are parallel, an *orthographic projection* of the sphere is obtained. The plane of projection being assumed perpendicular to the rays, constructions for obtaining the axes of the ellipses are deduced from the simplest principles of perspective and projection.



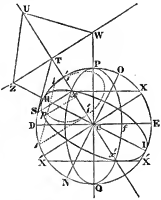
Let D P E Q be the primitive as before, P Q the intersecting line of a great circle perpendicular to the plane of projection; and let F G be the intersection of another great circle, with an auxiliary plane, which is perpendicular to the three former planes; this auxiliary plane being turned round on its intersecting line D E till it be brought to coincide with the primitive.

Draw F *f*, G *g* parallel to P Q, or perpendicular to D E, to cut D E in *f* and *g*; then P Q *f g* will be the axes of the ellipse, which is the projection of the circle (F G), and which is inclined to the plane of projection in the angle E C G, or D C F.

The half ellipse P *g* Q will not only be the projection of the half of the original circle on the other side of the plane of projection; but will also represent another great circle, inclined in an equal angle to this plane, but on the other side of the circle

P C Q, and having the diameter 2 G′ for its intersection with the auxiliary plane before mentioned.

If therefore the axis of the globe, as P Q, be in the plane of projection, the meridians are easily found by drawing lines parallel to P Q, through the points which divide the primitive into equal arcs, to cut E F; for all the meridians will have P Q for their major axis, and the segment of E F obtained by these constructions for their minor, and the parallels of latitude being, in this case, in planes perpendicular to the plane of projection, will be projected into right lines, as 1 1, 2 2, &c., drawn through the equal divisions of the quadrants.



Let P f Q be the orthographic projection of a great circle; from P and Q set off the arcs P X, Q X′ of any proposed number of degrees, and draw X x, X′ x' parallel to D E to cut the ellipse in x and x'; then P x, Q x' will be the projections of these arcs, and $x\,x'$ will be the projection of a diameter of the sphere. Draw x S perpendicular to $x\,x'$, to cut the primitive in S; draw a tangent to the primitive at S, to cut $x\,x'$ produced in T; make T U, in $x\,x'$ produced, equal to T S, and through T draw Z W perpendicular to T C for the intersecting line of a plane tangential to the sphere at the original of x. Produce Q P to cut this intersecting line in W, join W U, and make the angle W U Z of any given number of degrees: through Z and C draw a diameter, cutting the primitive in H and I, and draw N O perpendicular to H I. Then with H I for the major axis and x, x' for points in the curve (p. 251) find the minor axis in N O, of the elliptic projection of a great circle making the given angle W U Z with the former.

The same observation applies here, with regard to employing a circle of a larger radius than U T to set off the angle W U Z on, that was made on the analogous problem in the globular projection. (See p. 253.)

If x, x' be the poles of a series of parallel small circles, draw x S perpendicular to $x\,x'$, to cut the primitive in S; set off the equal arcs of a great circle intercepted by the originals of the small circles, from S, each way on the primitive as S 1, S 1, &c.; draw 1 1′, 1 1′ parallel to x S, to cut $x\,x'$ in 1′, 1′; join 1, 1, by a chord, and bisect it by a radius C S, or bisect 1′ 1′ in p; through p

draw a line perpendicular to $x\,x'$, or parallel to x S; set off on this line, on each side, the semichord 1 1, and the points thus found will be the vertices of the major, and 1′ 1′ will be those of the minor axis of the elliptic projection of the small circle.

By means of these constructions, the orthographic projection of the sphere (*fig.* 2, Pl. 8.) on the horizon of London is obtained; and from the figure the learner will see that, though this is more *natural* in appearance than the stereographic or globular projections, yet the portions of the spherical surface near the visible boundary, or the primitive, are so much *foreshortened* as to render this kind of projection ill calculated for geographical or astronomical purposes.

The next species of projection of the circles of the sphere that must be noticed is termed the *gnomonic;* in which the vertex is assumed as the centre of the solid, and the plane of projection as touching the spherical surface.

On this supposition, it is obvious that all great circles of the sphere will be projected into right lines, and the small circles will be projected into conic sections.

Let D E F, *fig.* 1, be a great circle of the sphere perpendicular to the plane of projection, C x the plane of projection, and D E, in

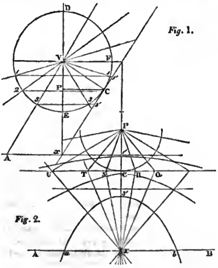


Fig. 1.

Fig. 2.

the plane of D E F, the common diameter of two or more great circles as meridians. V C will be the distance of the vertex;

draw C P perpendicular to V X; draw C x, *fig.* 2, and make it equal to C x in *fig.* 1; draw a line through C at right angles to C x, and make C P, in x C produced, equal to C P, *fig.* 1; from P, with P C for a radius, describe a circle, and make the angles C P S, C P R, S P T, R P Q, &c., equal to the angles of inclination of the planes of the great circles; then lines drawn through x and Q R, T U, &c., will be the projections of the great circles.

Let 1 1, 3 3, *fig.* 1, be the intersections of the planes of parallel small circles, having D E for their common axis, and being therefore perpendicular to the plane of D E F; produce 1 1, 2 2, 3 3, &c., to cut C x, and make C 1, C 2, C 3, &c., in *fig.* 2, equal to C 1′, C 2′, C 3′, &c.; draw lines through these points 1, 2, 3, &c., parallel to Q U, for the intersecting lines of the planes of the several circles.

Draw, in *fig.* 1, lines from V through the points 1, 1; 2, 2; 3, 3; &c., and transfer the segments in which they cut C x to *fig.* 2; also set off along A B, drawn through x at right angles to P x, the distances from x, that the same lines V 1, V 2, &c., cut A B in *fig.* 1. Thus make $x\,a$, $x\,b$, in *fig.* 2, equal x A, in *fig.* 1.

It will be immediately seen, that if any of the lines V 1, V 2, &c., in fig. 1, are parallel to the line C x, the corresponding small circle will be projected into a *parabola*, for the plane of projection will in that case cut the cone of rays in that curve; but all the other parallel circles will be projected either into ellipses or hyperbolas. If V 3 V 3 cut C x on the same side of C, then the curve of the projected circle will be an ellipse. If V 1, V 1, &c., cut C x on opposite sides of C, then the corresponding circle will be projected into opposite hyperbolas: and from the principles of perspective projection, guided by the knowledge of what each curve must be, the draughtsman will easily be able from the four points, as *a*, *b*, 3′ 3′ already got, to obtain other points in each curve, or other data for describing them; especially as one axis of all the curves must lie in P x, and therefore the other must be perpendicular to that line.

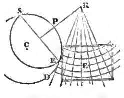
The gnomonic projection, when the axis of the globe is oblique, is rarely required, it being only used in dialling; but maps are constructed on this principle, the plane of projection being taken parallel to the axis of the globe, the meridians therefore being projected into parallel lines, while the parallels of latitude are projected into hyperbolas, the opposite curves forming the two parallels at equal distances on each side of the equator.

The Six Maps of the Earth, and of the Stars, forming part of the Atlas of the Society for the Diffusion of Useful Knowledge, are drawn on the gnomonic projection: in four of them the plane of projection is taken parallel to the axis; in the other two, the plane is assumed perpendicular to the axis of the earth at the poles, and consequently the meridians are projected into right lines, radii from the pole, while the parallels of latitude are projected into concentric circles.

The property of *development,*[*] of which cylindrical and conical surfaces admit, is very generally employed, in conjunction with the principles of projection, in the construction of maps of parts of the sphere.

If a small portion of the earth's surface only is required in a map, the easiest, and perhaps best mode of projecting it, is by the development of a right cone touching the sphere in the middle parallel of latitude of the part to be represented, and having its axis coinciding with that of the globe.

For example, let a portion of the surface of 30° in latitude, from 10° N. to 50° N. and 50° in longitude, be required to be delineated. Draw a semicircle to represent the half sphere, the diameter P C being

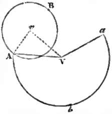


---

[*] There are certain species of curved surfaces which admit of a figure on a plane being made equal to them in area, by what is termed their *development*. *Conical* and *cylindrical* surfaces, which are supposed to be produced by the motion of a right line called the *generator* (see note, p. 207), always passing through a curve and a fixed point, or passing through a curve and always moving parallel to another fixed line, easily admit of development; for this generator may be considered, in all its consecutive positions, as a hinge on which every portion of the surface may turn till all be brought into one common plane.

If a cylindrical surface be cut by a plane perpendicular to the *generator*, the curve produced by the section will become a right line, when the surface is unrolled or *developed* into a plane. Hence if a straight line be made equal to the circumference of a circular base of a right cylinder, a rectangular parallelogram constructed on that line, having its altitude equal to that of the cylinder, will be equal to the surface of the solid. For if this parallelogram were rolled round the solid, one side being made to coincide with the *generator*, the parallelogram would exactly cover the surface of the solid.

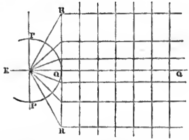
But no section of a conical surface, except the plane pass through the fixed point or apex, and therefore coincide with the generator, can develop in to a *right* line. The circular base of a right cone will, however, develop into an arc of a circle having the side of the cone for its radius, and the arc being equal to the circumference of the base, as is seen from the annexed figure, where the sector A V *a b* represents the developed surface of a right cone, having the circle A B for its base, and *v* V for its altitude; the arc A *b a* being equal to the circumference of the base.



Curved surfaces, which may be conceived as produced by the revolution of a *curve* round a fixed right line or chord, such as the *sphere*, the *spheroid*, *the paraboloid*, &c., which may be conceived as generated either by the revolution of a *circle* round a *diameter*, an *ellipse* round an *axis*, or a *parabola* round its axis, do not admit of development. A plane figure approximating to their surfaces may, however, be geometrically constructed, as will be subsequently shown of the sphere.

the axis; C D therefore, at right angles to P C, will represent the plane of the equator; divide the quadrant P D into degrees, and at the 30th degree draw a tangent, R E, to the semicircle, cutting C P produced in R. Draw radii from C through the proper divisions of latitude to cut the tangent. Make R E on the map equal to R E, and set off from E each way the distances at which the radii cut R E. From R, as a centre, describe arcs of circles through E, and the points thus marked in R E for the parallels of latitude. Draw the chord E S parallel to C D; describe a circle, anywhere apart, with E S for a diameter, and divide its circumference, or as much of it as may be necessary into single degrees, or into arcs of two, or three, or more, according to the size of the circle. Step as many of these divisions as may be necessary along the arc on the map through E each way; and then lines drawn through the proper divisions to R will represent the meridians. It is to be observed that the smaller the divisions of the circle, the more nearly will the arc described from R through E, along which they are stept, approach to the true development of that parallel of latitude.

If, instead of a cone, a right cylinder be supposed to touch the sphere at the equator, having its axis, therefore, coinciding with that of the sphere; the planes of the meridians will cut the cylinder in right lines parallel to its axis, and the parallels of latitude will be projected on the cylinder, from the centre of the sphere as a vertex, into circles equal and parallel to its base, or equal and parallel to the equator, these circles will develop into right lines perpendicular to the meridians, if the cylinder be supposed unrolled.



Let a right line Q Q, be made equal to the circumference of the equator E Q, and divided into thirty-six equal parts; then lines, perpendicular to Q Q, through the points of division, will represent the meridians as projected on the cylindrical surface, this being supposed to be developed; and if lines be drawn from the centre of the semicircle through the divisions in P Q P, representing the latitudes, these radii will cut the tangent R R in points through which lines being drawn parallel to Q Q, they will be the parallels of latitude, as is shown in the figure.

Mercator's projection, as it is called, employed in the construction of maps and charts, is analogous to the development of a cylinder as above described; but, for the particular purposes of navigation, it is requisite that the degrees of the meridians should be in the correct proportion to those of the corresponding longitude; and as the degrees of longitude are represented every where as equal, instead of diminishing towards the poles as they do on the surface of the sphere, the degrees of latitude, instead of being made equal to the tangents of the latitude as in the above development, are made to the corresponding degrees of longitude, in the proportion of radius to the cosine of the latitude.

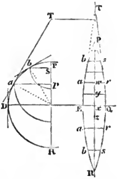
A table of the lengths of the arcs of the meridian in this projection, for every 10° of latitude, calculated from a formula investigated for the purpose, is to be found in all works on navigation; but as Mercator's projection does not properly come within the limits of *geometrical* projection, the following construction for an approximation to the true one is all that need be here given.

Describe the semicircle P Q P (see last figure) and divide each quadrant into nine equal arcs (of 10° each): draw the *secants* through the points of division, to cut the tangent R R. Then having stept the radius along any line Q Q, draw the meridians through the points thus marked, and make the segments of the meridian R R for the parallels of latitude, equal to the above *secants* of the arcs of 10° set off in succession; instead of equal to the tangents set off from Q Q.

Thus, by this construction, if the parallel through R represented the parallel of 60°; Q R, instead of being the tangent of 60° as in the figure, would be equal to the secant of 10° + secant of 20° + secant of 30° + . . . secant of 60°.

If a polyhedron of a great many sides be supposed to circumscribe the sphere, each face being tangential to it, then the development of this solid will be an approximation to that of the spherical surface; a substitute for this, practically used, will be first described.

Draw a right line P R, equal to the semi-circumference P D R, of a great circle of the sphere (Prob. 37, p. 67), and divide both the line and the semicircle into any number of equal parts; draw perpendiculars to P R, through the points of division: set off on the central one, E Q, half of one division each way to E and Q: *a p*, the next ordinate on each side of E Q, must be made equal to the same part of a right line equal to the semi-circumference of a small circle described on



the corresponding chord *a p* of the quadrant, and so of the next ordinate in succession each way, as *b s*, &c.; then curves being drawn through P *b a* E. .Q, and through P *s r*. .Q, the figure R E P Q will be an approximation to the development of a portion of the hemispherical surface, included between two great circles passing through P R.

If the semicircles were divided into eighteen parts, the two curves would be meridians, 10° apart developed, and the ordinates *a r*, &c. would be portions of the parallels of latitude included between them; E Q being a portion of the equator: and if thirty-six such figures had the corresponding portions of the map of the globe drawn on them, they would cover the surface of the sphere when applied to it. This is the mode employed by artificial globemakers for that purpose.

If the true development of the polyhedron were required as an approximation to that of the sphere, the following construction must be used.

Divide the quadrant P D into any number of equal parts in *a*, *b*, &c.; draw P R, E Q, at right angles to each other as before, cutting each other in *x*. Make *x* E, *x* Q, *x y*, *x z*, on these lines, equal to the tangent of half one arc of division, and draw lines through E, Q, *y* and *z* parallel to P R, and E Q, thus forming a square: set off the same tangent, each way along P R from *x* to *w*, &c., and draw ordinates through every other point of division, as *a r*, *b s*, &c. Make the half ordinate *a r*, equal to the tangent of the same angle, to the quadrant described on the corresponding semi-chord *a p*, of the circle: draw a tangent to the circle at *a*, to cut the diameter P R produced in T. Set off the length of this tangent, *a* T, from *w* to T in R T, and draw lines from T through *a* and *r*, which will cut the side of the square through *y* in its extremities*. Proceed in the same way with the tangent at *b*, and the corresponding ordinate *b s*, and so on till the last, when the sides of the triangles being drawn to P and R, they will complete the figure; P R containing the tangent of half the arc, as many times as the semicircle contains these equal semi-arcs.

This figure, though more correct in *principle*, will not be practically so correct as the former, for unless the number of divisions of the quadrant, and therefore of the sides of the polyhedron, be more numerous than can be *practically* employed, the area of this development will be very perceptibly greater than that of the corresponding spherical surface; while, by the former construction, P R being made in the ratio of 3·1416 to the semi-diameter, the difference between it and the real semi-circumference would be scarcely perceivable. But it is hardly necessary to observe that a plane can never be bent so as really to coincide with the spherical surface; and, accordingly, in either case, unless the lenticular developed figure be very narrow, the paper will *crease* on being applied to the globe.

---

* It might, therefore, appear unnecessary to find the semi-tangent *a w*, *w r*; but it is most important to have that semi-tangent correct, which it might not be if the lines drawn from the ongles of the square to T were alone trusted to.

If the vertex be out of the sphere, and a line passing through the vertex be carried round, always touching the surface, this line will generate a right cone, the axis of which will pass through the centre of the solid; and the small circle which constitutes its *base*, or that in which the line touches the sphere, will be the bounding outline of the solid, and is the *original* of the outline of the sphere projected as an object. If the plane of projection be perpendicular to the axis of the cone, this bounding circle will be projected into a circle, the original being parallel to the plane of projection. And the centre of the picture will be the projection of the centre of the solid, as has been mentioned in treating of the globular projection of the sphere. (See p. 253.) But if the plane of projection be not perpendicular to the axis of the cone of rays, the circle will be projected into a conic section, the axis of which will always pass through the centre of the picture.

For if a plane passing through the vertex perpendicular to the plane of projection, and therefore passing through the centre of the picture, also passes through the centre of the sphere, it will cut the plane of the circular base of the cone of rays in a diameter, which will be perpendicular to the intersecting line of that base; and the point which is the original of the centre of the conic section will lie in that diameter. (See figs. and demonstration, pp. 197, 198.) Consequently, the projection of that diameter, which is the intersection of the first-mentioned plane with the plane of projection, will also be a diameter of the conic section, which will pass through the centre of the picture, and will be perpendicular to the same intersecting line. Therefore, (Geom. App. Prop. 13) that diameter will be an axis.

Let C (Pl. 8, fig. 3) be the centre of the picture, *d e* the *projection* of any diameter of a sphere, which is parallel to the plane of projection, the point F bisecting *d e* will be the projection of the centre of the solid. Draw a line through F and C, and through any point F′ taken in this line at pleasure, but as far as convenient from F, draw P Q perpendicular to F′ C, and make

$F'P$, $F'Q$ each equal to Fd, or Fe: also draw a line ik through F perpendicular to F C, and draw C V parallel to ik, and equal to the distance of the vertex. (Def. 7, p. 125.) Join V F, the principal radial of ik considered as a vanishing line, F being its centre, and set off this principal radial each way from F along this vanishing line to i and k. By means of the three vanishing points F, i, and k, complete the projection, O S T U, of a square, its sides being perspectively equal to P Q or de. Bisect mn, the transverse axis of the inscribed ellipse, in o; through o draw the conjugate axis, and find its true vertices. (Prob. 55, p. 84.)

Find g, the vanishing point of lines perpendicular to the plane of the square O S T U (p. 159), and draw a vanishing line through g parallel to ik, or perpendicular to F C; draw Vg its principal radial, and set this off along the vanishing line each way from g; set off the length of the semi-conjugate axis above found from E along ik each way to d' and e'*, and complete the projection $w'x'y'z'$ of a square, having its sides perspectively equal to that conjugate axis, and parallel and perpendicular to the intersecting line of its plane. Then the ellipse inscribed in this square will be the projected outline of the sphere.

For the segment $d'e'$ of the vanishing line ik passing through F, is the *projection* of the circular section of the sphere by a plane passing through its centre and through the vertex (p. 126); and the plane (O S T U) is parallel to this circular section, both being perpendicular to the plane (FF'). And since P Q is made equal to de, P Q is the *plan* on the auxiliary plane (S U), of a diameter of the sphere which is parallel to the picture; consequently, the ellipse inscribed in the square O S T U, is the *projection* of the plan of the solid on the plane of that square produced by projecting lines parallel to the plane of projection, or this ellipse is the *projection* of the plan of the circular section of the sphere made by the plane (ik)†. The conjugate axis of this ellipse is the projection of the chord of the tangents from the station point of the original of mn (see fig. and demonstration, p. 197, *et seq.*), these tangents being projected into parallels to the transverse axis: hence this conjugate is the plan on the auxiliary plane (S U) of the intersection of the plane of the small circle constituting the visible boundary of the solid, with the plane (ik) to which the former plane, that of the base of the cone of rays, is perpendicular, as was before stated: the projection, therefore, of a circle on a diameter equal and parallel to that conjugate axis, and lying in a plane ($w' x' y' z'$) perpendicular to the plane ($i k$), will be the projection of the outline of the sphere as required.

---

* To simplify the figure, the given diameter of the sphere de is shown as coinciding with $d'e'$ in ik; but it must be understood that this is not necessary, and that de, $d'e'$ are not equal: the two latter letters are omitted in the plate.

† The section of the sphere by the plane (ik) is a great circle, and the projection of this by *parallel* lines on any plane as (S U) parallel to (ik), will be a circle equal to the former.

The intersecting lines of the planes not being required in this construction, the real magnitude of the sphere is neither given nor found, since the *same* ellipse or circle may be the projection of any number of spheres, provided these have the same right cone of rays touching the spherical surfaces in the visible outlines of the solids.

If the centre of the sphere were in the plane of projection, P Q would be the intersecting line of the auxiliary plane, and $d e$ would be the real diameter of the sphere. If the solid were between the vertex and the plane of projection, its real diameter would be less than $d e$, and greater if the sphere were beyond the plane. If any line, parallel to P Q, be assumed as the intersecting line of the auxiliary plane, then the segment of it intercepted by T S, V O, would be the real magnitude of the diameter $d e$.

§ THE PROJECTION OF SHADOWS.

If any luminous body be considered as a point, and the rays from it as right lines, the *shadow* of any geometrical figure produced by the rays from such a body is obviously the *projection* of that figure; and if the surface which receives the shadow be a geometrical one, such shadow may be found by the principles of perspective and projection.

But, since the figure itself must be represented on a plane by means of its *projection*, the shadow of it when drawn by geometrical rules is obviously the *projection* of a *projection*.

The delineation of shadows by geometrical rules is principally required in the determination of those cast by one part of a building, or machine, on another part, or on the floor or pavement; for the shadows of such objects on the rough surface of the earth are too undefined to admit of being delineated by rule; especially when it is considered that this branch of practical geometry is only required for the embellishment of a drawing, and is in no way necessary to its utility as a guide for the mechanic.

If the outline of an object be correctly drawn, the true form of it will be distinctly understood, even if the shadows of its various parts are not quite correctly represented; though any gross deviation from accuracy in this respect is fatal to a just conception of the original form; while, on the other hand, if an architectural, or other object be correctly drawn, and its shadow also truly projected, the illusion is complete*.

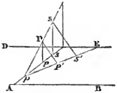
But besides the *positive* shadow cast by an object, it is necessary to be able to determine the limits of the illuminated parts: this presents *little* difficulty in the case of solids bounded by planes; a slight attention will show what faces of a regular solid, for example, will receive light from a given position of the luminous body: but a geometrical construction is necessary to determine this with regard to curved surfaces; and it is of such that it is most important to be correct in the delineation of their light and shade.

It is clear that the most complicated shadow, produced by a geometrical object on a geometrical surface of any kind, can only be a combination of such elements as the shadows of parallelopipeds, &c., cones, cylinders, spheres, &c., on planes or on curved surfaces; and these again may be further analysed into the shadows of plane right line figures, circles or plane curves, on such surfaces, which again are reducible to the shadows of right lines, or points.

---

* The architectural views exhibited at the Diorama may be cited as examples of this perfection, as far as outline is concerned. The French are far better draughtsmen than our countrymen.

If the object and the surface which receives the shadow do not fall within these limits, no geometrical constructions can be usefully employed, and the outline of the shadow, like that of the object itself, is better delineated by eye. And even in many cases, when the shadow can be determined by rule, the constructions are so complicated as to be useless, otherwise than as exercises for the improvement of the draughtsman, though far from being so, considered in this point of view.



1. If P be any point, and S any luminous body, the shadow of P, cast by S on any plane (A B), will be a point *p* in the intersection *S P*, or *S' P'* of *any* plane passing through S and P, with A B D E; if any two lines be drawn from P and S, parallel to each other in the plane, the points *P*, *S*, or *P'*, *S'*, in which they meet A B D E, will be in the same intersection. If, therefore, the points where two such parallel lines meet the plane on which the shadow is cast can be ascertained, then a line drawn through them will be cut by the ray drawn through the point and luminary in the shadow of that point.

2. The constructions for this purpose will be materially simplified if the parallels P *P*, S *S* be always assumed as parallel or perpendicular to the plane of projection: for in this case the projections of those parallels will be either parallel, and will have no vanishing point; or will have the centre of the picture for their vanishing point.

3. Let a plane be supposed to pass through the vertex perpendicular both to the plane of projection, and to the plane A B D E, this latter plane *not* being parallel to the plane of projection. The intersection of the two first planes will therefore be perpendicular to the intersecting and vanishing lines of the last; and will pass through the centre of its vanishing line (32, p. 143). If the lines P *P*, S *S*, be supposed not only parallel to the plane of projection, but also to this intersection, the determination of the points *P*, *S*, will be rendered still more easy.

Def. A line P *P*, or S *S* drawn to any plane A B D E from any original point P, or from any luminous point S, parallel to the plane of projection, and also to the line in which a plane through the vertex perpendicular both to the plane of projection, and to A B D E, cuts the plane of projection, is called the *support*[†] of that point P, or of the luminary S. And the point P, or S, in which the *support* meets the plane A B D E, is called the *plan* of P or S on A B D E.

4. If the plane A B D E be perpendicular to the plane of projection, the *support* of any point P will be perpendicular to the plane; but if A B D E be *not* perpendicular to the plane of projection, then the *support* is oblique to the plane: this is obviously the consequence of the *support* being in all cases *parallel* to the plane of projection. And the *projections* of the supports will always be perpendicular to the vanishing line of the plane.

Let A B be the intersecting, and D E the vanishing line of the plane; then if a luminous point S be at a finite distance from the plane of projection, its *plan S* will lie between A B and D E; because that plan is then at a finite distance from A B. But if S be considered as at an infinite distance, then its plan S will be in D E, since any point in D E is the projection of one at an infinite distance from A B, as the original of S must in this case be.

But the luminous point may also be either before, or behind, or in the plane of projection; its projection S may therefore have very different situations accordingly: this projection, however, can always be obtained, if the original be at a finite distance, by the simplest principles of projection. And if it be at an infinite distance, as the sun or moon, the principal luminaries by which shadows are cast may be considered to be with regard to any terrestrial object; then the projection S is obviously the *vanishing point* of the parallel rays from such a luminary, for the radial of these parallel rays is in that case the projecting line of the luminary itself.

If the luminary be at a finite distance, then in order to obtain its projection, its perpendicular distance from the plane of projection, and from any original plane, being given, assumed, or known, or any other sufficient data being had, its projection S is obtained like that of any other point in such circumstances.

Let C (Pl. 5, fig. 3) be the centre of the picture, and C V the distance of the vertex, X Y being the *horizontal line* (p. 189); draw the radial V S to make the angle C V S equal to that which a plane passing through the vertex and the sun perpendicular to the horizon makes with the plane of projection; then S is the *plan* of the sun. Set off the length of the radial V S, from S to *v*, in Y X; and make the angle S *v*[*] equal to the *altitude* of the sun, either given or assumed; draw S* perpendicular to X Y, to cut v* in *, which will be the *projection* of the sun on the plane of projection, or the vanishing point of the parallel rays. If the sun be behind the plane of projection, * will be *below* the horizontal line, if *before*, * will be *above* the horizontal line. If the sun were in the *vertical plane* (Def. 3, p. 125) C V S would be a right angle, and the rays being parallel to the plane of projection, could have no vanishing point.

[†] This term is introduced to avoid the perplexing repetition of the words *projecting*, *projection*, &c., for it is obvious that the *support* is in fact the projecting line of the point with regard to the plane A B D E: but as the point has another projecting line, by which its projection on the plane of the picture is determined, the word support is used instead.

The point which is the *projection* of the luminary on the plane of projection, being obtained by these principles, it will always, in future, be simply called the luminary,[†] and will be assumed as given; and since any original points, lines, or objects, are only capable of being represented by their projections, and the mode of obtaining these having been explained in the preceding part of this work, in future, in treating of shadows, such projections will be assumed as given or found, and will be called the *original* points, lines, &c.

And since, in future, all considerations of the mode of obtaining these projections will be omitted, unless particularly necessary to elucidate subsequent constructions relating to their shadows, the term *ray* will be applied to the line from the luminary through a point, by which its shadow is determined, and the plane passing through a luminary and an original line will be called the *projecting plane* of that line; while the plane on which the shadow is cast will be called the *plane of the shadow*.

The following general theorems relating to the vanishing lines and points of original lines, their plans, rays, and shadows, will be found necessary for the illustration of this subject.

5. Let E D (Pl. 8, fig. 4) be the vanishing line of any plane, * any luminary, l its *plan* on the plane [D E], O, P, any two points in the original line O P, which line meets the plane in Q, and o, p the plans of O and P; then

Def. A line, $o\ p$, drawn through the *plans* (Def. p. 266) of any two points in an original line, is called the *plan* of that original line.

6. The *plan* $o p$ of an original line will obviously pass through Q, the point in which the line itself cuts the plane in which that plan lies. Because, the lines O P, $o p$ are both in one plane passing through the *supports* of O and P; Q, therefore, their common intersection, must be in that of the planes passing through the two lines.

7. The intersecting line of the plane [O o P p] will be parallel to the supports O o, P p; unless the original line O P is parallel

---

[†] It will be always expressed by a * in the text and figures, and its *plan* on any original plane will be marked l, if it be at a finite distance, or s if it be the sun, &c.

to the plane of projection; for those supports are parallel to the plane of projection (Def. p. 266), and therefore will be parallel to the intersecting line of the plane passing through them. (4. p. 126.)

8. And since the supports O o, Pp are parallel to the line which is drawn perpendicular to the vanishing line D E through its centre, the intersecting line of the plane [Op] will be perpendicular to the same vanishing line.

9. Since the plan op lies in the plane [D E], D, in which op cuts D E, must be the vanishing point of op (18. p. 131); and if Dx be drawn perpendicular to D E, or *parallel* to the intersecting line of the plane [O p] in which O P lies, then x will be the vanishing point of the original line O P. For D being the vanishing point of the intersection of two planes, it must be the intersection of the vanishing lines of those planes; D x therefore, drawn parallel to the intersecting line, must be the vanishing line of the plane [O p] in which O P lies; consequently x in that vanishing line must be the vanishing point of O P.

10. $o\,l$, $p\,l$, are obviously the *plans* of the *rays* of the points O, P, consequently for the foregoing reasons, if $y\,z$ be drawn through the vanishing point of the *plan* $p\,l$, *perpendicular* to D E; or *parallel* to the intersecting line of the projecting plane [* l P], $y\,z$ will cut the ray * P in z, the vanishing point of that ray.

11. If the luminary were at an infinite distance, l and y would coincide, as has been before explained (p. 267), and therefore * and z would also coincide.

12. For the same reason that op, the *plan* of an original line O P, must pass through Q, the shadow of O P must pass through Q; consequently, if either o' or p', the shadow of any one original point as O or P, be found, a right line (Geom. IV. Prop. 2) drawn through it and through Q must be the shadow sought; and will pass through the shadow of the other point. Now o' or p' is the intersection of the ray of the original point with the *plan* of that ray.

13. If O P be parallel to the plane [D E], it will be parallel to its plan; and will therefore have a common vanishing point; that is, in that case Q and D will coincide, and the shadow $o'\,p'$ would also have D for its vanishing point, because that shadow would be *parallel* to the original line. (Geom. IV., Prop. 10.)

14. Draw $l\,m$ parallel to D E, to cut o D in m; and draw $m\,n$ perpendicular to D E, to cut the original line in n; draw the ray * n. Then the intersecting and vanishing lines of the projecting plane of O P will be parallel to * n. For $l\,m$ is parallel to the plane of projection, as is also $m\,n$ and * l; therefore the ray * n lying in the plane [* $l\,m\,n$] is parallel to the same plane, and con-

sequently must be parallel to the vanishing and intersecting lines of the projecting plane, in which * n lies. The vanishing line therefore of this projecting plane will be a line drawn through x, parallel to * n.

15. The lines $l\,m$, * n will cut each other in a point of the shadow of O P, because n being an original point in the line O P, its shadow must be in that of the line O P in which n is situated.

16. If a line l D be drawn through l, the *plan* of the luminary, and through D, the vanishing point of the plan of an original line, and if a ray * x be drawn through the luminary, and through x, the vanishing point of the original line; then these two lines l D, * x, will intersect in a point a, through which the shadow of the original, and of all lines having x for their vanishing point, will pass. For, on account of the vanishing line D x, to which * l is parallel, l D, * x are in one plane; but l D lies in the plane [D E], and the ray * x lies in the projecting plane of the line O P, consequently their intersection a must lie in that of these two last-mentioned planes, which intersection is the shadow of the original line; or a lies in that shadow. Now all other lines having x for their vanishing point will have D for that of their plans on [D E]; therefore the points x, D, and * remaining the same, a will be constant, for every original line passing through x; the shadows, therefore, of all such original lines will pass through a.

If the luminary were at an infinite distance, l would be in D E; consequently a would be also in D E, or would be the point in which the ray * x cuts that line.

If the original line were parallel to the plane of the picture, its *plan* would be parallel to the vanishing line D E of the plane of the shadow, and neither it, nor its plan, could have a vanishing point, that is, x and D would be at an infinite distance; * x must in that case be drawn parallel to the original line, and l D parallel to the vanishing line D E of the plane of the shadow, and the shadow of the line would pass through a, in which those lines * x, l D, met.

If the original line were parallel to the plane of the shadow, as well as to the plane of projection, * x would be parallel to the original line, and therefore to D E the vanishing line of the plane of the shadow; consequently it would be also parallel to l D; a would therefore be at an infinite distance, or the shadow of the line would be also parallel to the original, and to its plan: the truth of this conclusion is also obvious from the principles of projection.

These modifications of the last theorem required explanation, from their practical importance in the projection of shadows.

Let E G be the vanishing line of another plane, on which it is required to find the shadow of the same original line O P, as cast by the same luminary *.

E F being drawn through E, and through the point in which the intersecting lines of the planes [F G, D E] cut one another, E F will be the intersection of the two planes (19, p. 131). Draw lines parallel to D E, the vanishing line of the first planes, through *l*, *o*, and *p*, the plans of the luminary, and of any two points in the original line, as given or found on that plane (D E); then draw lines parallel to E G, through the points *q*, *q*, in which these parallels cut E F, and these last parallels will cut perpendiculars to E G, drawn from the luminary and the two original points in the *plans* *L*, *O*, *P*, of the luminary, and those points on the new plane (E G).

For the *supports* on the new plane, of the luminary and of the original points, will be perpendicular to the vanishing line of that new plane (4 p. 267); and as these new supports, like the former, are parallel to the plane of the picture, a plane passing through the two supports of any the same point, as P, will also be parallel to the picture, and will therefore cut all planes in lines parallel to their intersecting lines, consequently *p q*, *q P* are the intersections of such a plane passing through P, with the two planes [D E, E G]; *P* therefore must be the plan of P on the plane [E G].

If L O, L P, the plans of the rays, on the new plane, be drawn, they will cut the rays of the points in their shadows (1. p. 266) o'', p''; $o''\,p''$ being drawn is therefore that of the original line on the new plane as required.

If *O P*, the plan of the line, be drawn, it will cut the original line O P in T, the point of intersection of the original line with the new plane, and the shadow $o''\,p''$ will pass through T. $o''\,p''$ will also be found to pass through a point (a'),[†] if found, as before, by drawing a line through *L*, and the vanishing point in E G of *O P*, to cut the ray * *x*.

All other theorems which were demonstrated relative to the shadow of O P, on the plane [D E], will, of course, equally apply to the new plane [E G], because these theorems were totally independent of the angle in which the plane of the shadow was inclined to the plane of the picture, and are therefore generally applicable to all planes.

If *y p* (Pl. 7, fig. 2) be an original line lying in a plane (D A), its shadow, as cast by a luminary * on another plane (D K), may be found by the following construction, also of general application, whether the luminary be at a finite or at an infinite distance. Draw a line *s t* (*l t*) parallel to the vanishing line E C of the second plane (D K) to cut another line * *t* drawn parallel to the vanishing line E R of the first plane (D A) in the point *t*; also draw *y p*, *z q* through any two points in the original line, parallel to E R, to cut the intersection D E of the two planes in points *p*, *q*. Then lines drawn from *t* through *p* and *q* will cut the rays * *y*, * *z* of the two original points in *y'*, *z'*, the shadows of those points; and a line *y' z'* drawn through these shadows will be the shadow of the line on the plane (D K).

[†] The point specified as a' in the text does not fall within the plate.

For * *s* (* *l*) the support of the luminary on the plane (D K), and *s t* (*l t*), being both parallel to the plane of the picture, * *t* is also parallel to it, as is likewise *y p* (26. p. 132); consequently *t p*, and the ray * *y* lie in one plane, which intersects the plane (D K) in *t y'*; therefore *y'* in that intersection must be the shadow of *y* (1. p. 266). In the same way *z'* may be shown to be the shadow of *z*, and *y' z'* is therefore the shadow of the original line on the plane (D K).

When the luminary is at an infinite distance, its plan *s* being in that case in the vanishing line E C, *t* will be in the same line, for *s t* will coincide with that vanishing line.

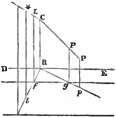
Def. A line * *t* drawn from the luminary, or a line drawn from any original point in an original plane, parallel to the vanishing line of that plane, till it cuts the plane of the shadow, is called the *parallel support on the plane of the shadow with respect to the plane of the line*, of the luminary, or that original point; and the point in which this parallel support meets the plane of the shadow is called the *parallel plan* of the luminary or point.†

If the plane of the shadow be parallel to the plane of the picture, then the *supports* of the luminary, and of original points, must be *perpendicular* to both planes, and these supports will be projected into lines having the centre of the picture for their vanishing points.

Let C be the centre of the picture, and D E the vanishing line of a plane, as before, the intersection *f g* of a plane parallel to that of the picture with the plane (D E) will be parallel to that vanishing line: draw * C, P C from the luminary, and any original point to the centre of the picture; and from *l* and *p*, the plans of the same points, draw lines to R, the centre of the vanishing line D E, cutting *f g* in *f* and *g*; then *f L* drawn perpendicular to D E, or parallel to R C, will cut * C in *L*, the plan of the luminary on the new plane; and in the same way, *P*, the plan of the original point on the same plane, is found by drawing *g P*

---

† These terms are necessary to distinguish these lines and plans from the ordinary supports and plans already defined. (Def. p. 266.)

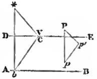


perpendicular to D E. For the points *L* and *P* are the intersections of the new plane by the supports * C, P C; since *f L*, *g P* obviously lie in that new plane, and in planes perpendicular to it, and to the plane of the picture, and passing through the luminary and original point.

Having obtained the plans of the original lines and luminary, the mode of obtaining the shadows of them is precisely the same as that already given.

If the luminary be at an infinite distance, the plans of the rays on a plane parallel to the picture will be parallel to a line drawn through the luminary *, and through C; for in this case, unless the line passing through the vertex, perpendicular to the picture, also pass through the infinitely distant luminary, when C will of course represent both its plan on the plane, and the luminary * itself, there can be no plan of it, because the plane has no vanishing line in which such a plan would be situated (4. p. 267); therefore a line * C drawn as above, is the vanishing line of a plane passing through the luminary, and through the support of any point P; consequently a line from P, parallel to * C, will be the intersection of such a plane with the original plane which is parallel to the picture, and this intersection is the plan of the ray.

There is one situation of the original luminary when at a finite distance, which requires a particular construction, in order to find the shadows projected by it; this is when the luminary is in the vertical plane (Def. 3. p. 125). Because, in that case, its plan on any original plane, not parallel to the picture, must also lie in the vertical plane, and consequently neither can have any projection on the picture from which the rays, or their plans, can be drawn.



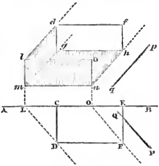
Let P *p* be the support of a point P, the shadow of which, cast by a luminary situated in the vertical plane, is sought. Consider A B, D E as representing the station line and parallel of the vertex (Def. 13, p. 130), instead of the intersecting and vanishing line, and C will, on this supposition, represent the vertex. If * be the situation of the luminary in the vertical plane, draw * *b* perpendicular to A B; then *b* will be the plan of the luminary on the original plane; draw *b c*, * *c*, and *p p'*, P *p'* parallel to them, cutting in *p'*, *p'* will be the shadow required.

For *b*, *, being supposed as lying in the vertical plane, these points are the station points of the plans of the rays, and of the rays themselves; therefore V *b*, V * are the verticals (Def. 15, p. 134) of those plans and rays, and are parallel to the projections of those lines (29. p. 134); hence *p p′*, P *p′* are those projections, and *p′*, their intersection, is that of the shadow, the paper being again supposed to be the plane of the picture.

This is sufficient to explain the analogous mode of proceeding in such cases, as variously modified by different positions of the original point and plane of the shadow, but as these are rare of occurrence in practice, this particular instance does not demand any further explanation.

An infinitely distant luminary, through which the vertical plane is supposed to pass, has been already noticed. (See p. 268.)

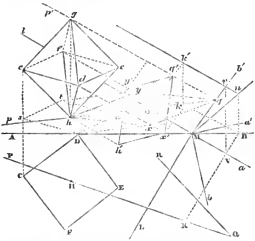
If the original object be projected *orthographically*, instead of perspectively, the practical application of the foregoing principles will be modified accordingly: an example of the projection of the shadows of plane geometrical solids, represented in both modes, will now be given, with the necessary observations on each.



Let C D E F be the plan, and *d f g h* the elevation of a rectangular parallelopiped, one face touching the perpendicular plane (A B); the line P Q, *p q* being the plan of the solar rays as projected on the two planes perpendicular to each other. Through *d*, *f*, *g*, *h*, draw lines parallel to *p q*, as *d l*, *g m*, *h n*, and through D draw a line parallel to P Q, to meet A B in L; through L draw a line parallel to *d g*, to cut *d l*, *g m*, in *l* and *m*. Draw M *n* parallel to *g h*, then *d l m n h* will be the shadow of the solid cast on the plane (A B) by rays parallel to P Q, *p q*. In this way the shadow of any rectangular projecting part of a building, such as the cill of a window, &c., may be determined in an architectural *elevation*.

If, instead of being the orthographic projection of a solid, *d f g h* were a rectangle, parallel to the plane (A B), its shadow would be the equal rectangle *l m n o*, and this is obtained in the same way; for *d l*, as a line, is obviously the shadow of C D, *d* considered as a line; and *l* is the shadow of the point D, *d*; for the same reason *l*, *m*, *n*, *o*, are the shadows of the four angles of the parallelogram. The rectangle of the shadow is equal to the original, because that shadow is the orthographic projection on a plane parallel to that original, (47, p. 152, and p. 231.) the rays being parallel.

c d e f g h is the *oblique* orthographic projection of an octohedron, the diagonal *g h* being parallel to the plane of projection and perpendicular to the plane (A B), which is to receive the shadow. P Q is given, or assumed, as the horizontal



projection of a ray; *p q* being the oblique projection of P Q, and P Q R being the angle which the ray makes with the horizontal plane: then the angle *p q p'* being made the *projection* of P Q R by the principles of projection, *p' q* will be the oblique projection of a solar ray; draw *p q* through *h*, parallel to the projection of P Q †, and draw *p' q* through *g*, parallel to the projection of the ray just found; *q* will therefore be the shadow of the apex *g* of the solid, or *h q* will be the shadow of the diagonal *h g*.

Draw the rhombus D *s t u*, equal and similar to *d c f e*, that is, draw the *plan* of the solid on the plane (A B) of the shadow. Then the points *s*, *t*, *u*, will be the *plans* of the points *c*, *f*, *e* on the plane of the shadow. Draw lines parallel to *p q* through D, *s*, *t*, and *u*, and lines parallel to *p' q* through *d*, *c*, *f*, and *e*, cutting the former in *x*, *y*, &c. respectively, which points will obviously be the shadows of the corners of the solid, and hx, hy, qx and qy being drawn, $hxqy$ will be the shadow of the solid on the plane (A B). The shadows of the points c and e falling within the outermost lines bounding the outline of the shadow, it is clear that the four faces hde, hef, gde, gef, are in shade; and the other four receive and intercept the rays, or are illuminated; consequently the shadow is that of the section $hdfg$ of the solid.

---

† That is, draw *p q* through *h*, if P Q the plan of the ray, and *p q* its oblique projection, are not given as passing through H and *h*; which may not be the case: P Q, *p q* were shown as passing through these points in the figure to avoid unnecessary lines.

Instead of the plane (A B) receiving the whole shadow, let another plane be supposed to intercept part of it, L M being assumed or given as the intersection of this plane with the first, and the angle a M b as its inclination to (A B). Having found O M, the *oblique* projection of L M, draw the angle O M a', the oblique projection of the right angle L M a, and the angle a' M b' that of the angle a M q, by the principles of projection.

Draw a line K B at pleasure parallel to L M, to cut A B in B and M a in V; draw V v, B n perpendicular to A B, V v cutting M b' in v; or draw B k parallel to M l, to cut hq in k; then draw kk' perpendicular to A B, and make kk' equal to V b; through the points k' and v, draw $k'n$ parallel to M O, then $k'n$ will be the oblique projection of the original line, of which B K is the *plan;* and M n, drawn through n, in which lie $k'v$, B n, intersect, will be the *intersecting line* of the second plane.

Join O k', then the point q', in which this line is cut by the ray $p'q$, will be the shadow of g on the new plane; draw a line parallel to O k', from the point in which ty cuts M O, then y', the point in which this parallel is cut by the ray fy, will be the shadow of the angle f on the new plane. If lines be then drawn from q' to y', to the points in M O, where xq cuts M O, and from y' to the point in M O, where hy cuts M O, they will complete the portion of the shadow intercepted by the new plane.†

If k' O were produced to cut the ray hh' in h', h' would be the shadow of the apex h on the second plane; produce D x to cut M O in a point, through which a line being drawn parallel to O k', it will cut the ray dx in x' the shadow of d on the second plane; $q'x'$ being drawn will then cut M O in the same point that xq cuts it in: $h'y'q'x'$ will be the shadow of the octohedron on the new plane, supposing the first (A B) removed.

If the diagonal gh of the solid had not been parallel to the plane of projection, or perpendicular to the plane (A B), the proceeding would be the same in principle: the oblique projections of the *plan* (Def. p. 266) of the apex g, and of the other angles of the solid, as c, d, e, &c., must be first found, then lines being drawn through these *plans*, parallel to pq, the *plan* of the ray, they will be cut by the respective rays of the points, in the shadows of those points as before; for every such shadow will lie in the plane passing through the luminary, and through the *support* of the point, and this plane will cut the plane of the shadow (A B) in a line parallel to $p\,q$, the luminary being by supposition at an infinite distance.

---

† In the figure the tint of the shadow is omitted near the angles y' y' to preserve these points distinct. it must be supposed as carried on to the portion of $h'y'$ cut off by M O.

If $g\,h$ be considered as the *support* of g, then the *projecting plane* (see p. 268) of the point g will cut the plane (A B) in $h\,q$, drawn parallel to the plan of the ray; therefore q is the shadow of g. In order to find the shadow of this same point on another plane, as M O $v\,k'$, it is first necessary to find the intersection of the same *projecting plane* of g with this new plane: now this intersection will obviously pass through O, M O being that of the two planes: to find O k' it is therefore necessary to construct the projection O $k\,k'$ of the triangle lying in the *projecting plane* of g, and formed by the intersections with it; 1st, of the original plane (A B); 2nd, of any plane assumed at pleasure, perpendicular to (A B), and cutting this in a line parallel to M O (M L); and 3rd, of the new plane of the shadow.

The auxiliary plane just mentioned will cut the new plane of the shadow in a line parallel to M O (M L), because B k (B K) is assumed parallel to M O(M L); and all perpendiculars to B k (B K), lying in the auxiliary plane, and meeting this intersection of the two planes, will be equal to V b, (V M b being the angle of inclination of the planes (A B) (M L) of the shadow, and the triangle V M b being supposed turned down on M V till it coincide with (A B).) And as all these perpendiculars are parallel to the *support* $g\,h$, and therefore are parallel to the plane of projection, their oblique projections as $k\,k'$ will be equal to the originals.

L M . . . T W (Pl. 5, fig. 3) is the general outline of a building perspectively projected, * being the sun (see p. 267). To find the shadow of the building on the ground, draw lines from S, the *plan* of the luminary through L, M . . . . R, the *plans* of the angles of the building; which plans, from the symmetry of the object, will coincide with the bottoms of the upright edges of the walls; these lines S L, S W′, S M, &c., will be cut by the rays in the shadows of the respective points, and L $o\,w\,n\,u$ R will be the shadow as thrown towards the spectator, the sun being beyond the plane of projection; and L $o'\,t'\,w'\,u'$ R will be the shadow if the luminary be on the same side of the plane of projection with the spectator.

In the former case the wall L O S T, and the face O W T of the roof will be in light; though the rays of light will fall very obliquely on the wall, as will be perceived from the acute angle made by S L, with L Y (the angle S V Y); the wall L O S T receiving the light, the parallel wall M N R U, and the face U N W of the roof will be in shadow.

In the latter supposition of the situation of the sun, L O S T O W T will be in shadow, and M N R U will receive the light, the rays falling on it at the same oblique angle as they fell on L O S T.

The faces O W N, T U W, will be in shadow respectively, in both positions of the sun, as will be perceived from the circumstance of the shadow of the apex W being without the shadow $o\,n$, of O N, and without the shadow $t'\,u'$ of T U.

Since the original lines N O, O T, T U, U N, are parallel to the plane of the shadow, the shadows of those lines will be parallel to the originals (13, p. 269); these shadows therefore will have X and Y for their respective vanishing points.

If the luminary were in the vertical plane, or in the plane of projection, then the lines $L\,o''$, $W\,w''$ &c., must be drawn parallel to the vanishing line X Y; and the rays $O\,o''$, $W\,w''$, &c., must be drawn parallel to each other, and to form an angle with that vanishing line equal to the altitude of the sun (see pp. 267, 268). In this case the walls L M N O, M R N U, and the faces O W N, N W U will receive the light, and the others will be in shade.

If the height of the apex W, of the roof above the plane N O T U, had been less; so that the shadow w, of that point, had fallen within the lines $o\,n$, or $t'\,u''$, or $o''\,t''$, then all four faces of the roof would have received the light; the same effect would, of course, have been produced if the *altitude* of the sun had been greater, so as to cause w to fall within those lines.

C being the centre of the picture (Pl. 5, fig. 4), and the luminary * being supposed at a finite distance, the shadow of the rectangular parallelopiped L O N . . R T, on a plane (R T) perpendicular to the horizon, is found in precisely the same manner, the *plan*, l, of the luminary on the plane of the shadow being first obtained. (See p. 272.)

If, in this example, * were the projection of the sun, behind the spectator, there could not be found any point l for its *plan* on the plane of the shadow (see p. 273); in this case lines must be drawn through R, U, S, the *plans* of N, M, L, parallel to * C, for the plans of the rays, and these will be cut by the rays, as before, in the shadows of the points.

If * and C coincided, which would be the case if the radial, perpendicular to the picture, also passed through the sun, then there would be no line * C to which the *plans* of the rays could be drawn parallel, consequently the shadows of the points L, M, N, O, would coincide with their *plans* S, R, U, T; this conclusion is obvious, because each of the *supports* L S, N U, &c., would also pass through the sun; in this case the parallelopiped would cast no shadow on the plane of its face (R T).

These examples, if thoroughly well studied, will be sufficient to enable the learner to draw the shadow of any right line figure, or plane solid, on any plane, as cast by a luminary in any position, and at a finite or infinite distance, whether the object be orthographically or perspectively represented.

The shadows of curve lines, being *projections* of those curves, admit of the same observations as have been made on the subject of the simple projections of such lines.

The shadow of a circle in any position, the luminary not being in the plane of it, will be a conic section, if the shadow be received on a plane; and the species of curve which will be the shadow of a given circle, depends on the relative position of the circle, the luminary, and the plane of the shadow. (See pp. 196 and 204.)

The shadow of the circle, and of all other curves, is best found by obtaining the shadows of a sufficient number of points in the curve, and then drawing the required shadow through them.

But, as has been before remarked, it is necessary not only to obtain the shadow of solids with curved surfaces, but also the line separating the light from the shade on those surfaces. The mode of proceeding with the cylinder, as the most important of such solids, will now be explained.

If two planes be supposed to pass through the luminary, and to be tangential to the cylindrical surface, the two straight lines in which the planes touch the cylinder will separate the illuminated part from that in shadow; and the intersections of these tangent planes with the plane of the shadow will be the outlines of the shadow of the cylindrical surface, or will be the shadows of the two straight lines above mentioned. The common intersection of the two tangential planes will also be a line passing through the luminary parallel to the two lines on the cylindrical surface, and therefore to the axis of the solid (see p. 207). If these three parallels be produced to cut the plane of the shadow, then lines drawn from the point, in which that passing through the luminary cuts the plane, through the two points in which the other parallels cut the plane, will be the straight outlines of the shadow of the solid. Let the luminary * (Pl. 7, figs. 2 and 3†) be assumed as at an infinite distance, *s* being its plan on the plane of the shadow (D K, C E), and *s*′ its plan on the plane (A D, E R) of the base of the solid; draw * S, which line will represent the common intersection of the two tangential planes; for S being the vanishing point of the axis of the solid, (see the constructions for drawing the projection of this cylinder, p. 216, *et seq.*) all lines drawn through S will represent lines parallel to that axis (9. p. 128); and since both S and * represent points infinitely distant, the original of * S is so, and therefore can only cut the plane of the shadow in the vanishing line C E of that plane;† consequently r represents the point in which * S cuts the plane of the shadow.

---

† The points and lines which did not fall within the plate in fig. 3, are shown in fig. 2 to a reduced scale, and reference must be made to this figure whenever the luminary or its plans are mentioned in the text.

Draw $r\,y'$, $r\,z'$, from r, tangents to the elliptic section of the solid by the plane of the shadow; draw lines to S from w and x, the points of contact of the curve and the tangents, to cut the circular base of the solid in y and z: then $w\,y$, $x\,z$ will separate the light from the dark part of the cylindrical surface; and rays * y, * z drawn through y and z, will cut $r\,y'$, $r\,z'$ in y' and z'; $w\,y'\,x\,z'$ will therefore be the straight part of the outline of the shadow. It is obvious from the parallelism of the tangential planes in this example, that $y\,z$ will be a diameter of the circular base, and will therefore pass through c.

If $y\,z$ be produced to p, its vanishing point in E R, then a ray drawn from * through p, will cut E C, in the vanishing point of the shadow $y'\,z'$ of the diameter $y\,z$. (16. p. 270.)

The vanishing point p, of the chord or diameter $y\,z$, may be found at once by the following construction. Draw V r', the radial of the point r', in which * S cuts the plane of the circular base; then V p drawn perpendicular to V r' will cut the vanishing line in p, the vanishing point sought. For by this construction, $y\,z$ represents a line perpendicular to one drawn from r through c the centre of the circular base: that is, $y\,z$ is the chord of the tangents to the point in which * S cuts the plane of that base; those tangents $r'\,y$, $r'\,z$, being the intersections of the tangential planes above mentioned, with the plane of the base.

The shadow $y'\,z'$ may also be found by the method explained (pp. 271, 272) by means of t, the *parallel plan* of the luminary. (See Def. p. 272.)

The next step to complete the shadow of the solid is to find that of the circular base, or as much of it as forms a part of the whole shadow of the solid; and this will obviously, in the present instance, be the shadow of the semicircle $y\,f\,z$. Draw any lines at pleasure, parallel to A D, to cut the circle, and produce them to cut D E; then a line drawn from t, the *parallel plan* of the luminary, through the point of intersection in D E, will be cut by the rays of the extremities of the chord, in the shadows of those extremities; and will therefore furnish points in the curve of the shadow: this construction is shown in the figure as made for a diameter through c, and the principle has been already explained. (See pp. 271, 272.)

---

† This will be rendered still clearer to the learner by the following considerations: If * S could be supposed to cut the plane of the shadow in a point *not* being in the vanishing line of that plane, no lines drawn from such a point could represent *parallel* lines lying in the plane: now the two tangential planes above mentioned, touching the surface and passing through the luminary, must be parallel to each other, if this luminary be infinitely distant, (unless the axis of the cylinder were supposed to pass through it, which is not the case, or the solid could cast no shadow on a plane cutting the cylinder). These parallel planes will cut the plane of the shadow in two *parallel* lines, which must be projected into lines having a vanishing point in the vanishing line of the plane of the shadow; therefore, the point representing that in which * S may be supposed to cut the plane of the shadow, must lie in the vanishing line, or cannot lie out of it.

The usual mode of finding the shadow of any point may be employed for the same purpose. Find the *plan* on the plane (D k, E c) of the shadow, of the extremities of the chords drawn at pleasure to cut the circle; then lines drawn from r through these plans will be cut by the rays in the shadows sought. In the figure, the *support* of one extremity of the diameter through c is shown as well as the method of determining the *plan* of that extremity, by drawing a line parallel to D K, through the point in D E in which the diameter produced cuts D E.

When a sufficient number of points in the outline of the curve of the shadow have been found by these constructions, the curve of the ellipse may be drawn by hand, and the lines $w\,y'$ $x\,z'$ will be tangents to the curve at y' and z', because the *projecting planes* (p. 268) of the lines $w\,y$, $x\,z$ are tangential to the original circle.[†]

The shadow on the same plane (D K, E C) of the other cylinder, having its axis $y\,z$ parallel to the plane of the picture, is obtained by the same principles.[‡] Lines drawn from s (fig. 2) tangential to the circular bases of the solid, at w, x, p and q, determine the lines $w\,p$, $x\,q$, on the cylindrical surface, which separate the illuminated half of that surface from the half in shade. The shadow of the diameter $p\,q$ will be parallel and equal to the original, because $p\,q$ is parallel to the plane of the shadow; consequently the shadow will have a common vanishing point in E C, with $p\,q$. This vanishing point might be found in the same way as that of $y\,z$ was for the other cylinder; by drawing the radial of that vanishing point perpendicular to the radial of the point s.

Points in the shadow of the circle $q\,p\,f\,q$ are found in the same manner as above explained for the other cylinder; and in this case, since the *supports* of all points in that circle lie in the cylindrical surface, the *plans* must lie in the base of the cylinder lying in the plane of the shadow, consequently the constructions are much simplified.

[†] It is obvious that $w\,y'$, $x\,z'$ are the shadows of the tangents at y and z drawn from r', and therefore lying in the plane of the circle: hence also the elliptic shadow is touched by $w\,y'$ $x\,z'$ in y' and z'. (Geom. App. Prop. 6.)

[‡] It will be easily perceived that the *support* of the luminary * s, is analogous to * S in the last example: that is, * s is a line parallel to the axis of the cylinder; and in this case the *projections* of that line and the axis are really parallel, because both originals are parallel to the plane of the picture, or the vanishing point (S) of the axis is at an infinite distance.

Since the cylinder (or cone) of rays, which produce the shadow of a circle, is cut by the plane of the shadow to produce the shadow, the rays which form the outline of the cylinder of rays must be tangents to the curve of the shadow (see p. 208): this rule affords an additional guide for drawing the curve of the shadow, as is shown in the present example by the ray near the point *s* in the circle *p g s*, this ray being also a tangent to the curve of the shadow.

The modifications of the foregoing constructions, which would be necessary if the luminary were at a finite distance, will not require to be entered into, since they must be obvious to an attentive reader of what has been just explained; nor can he find any difficulty in drawing the shadow of a *cone* as cast by any luminary, if he make himself master of the principles relative to the projections of that solid. (p. 208, *et seq.*)

For, if the shadow of the apex of the cone be obtained, then lines drawn from that shadow, tangents to the shadow of the base, or tangents to the section of the conic surface by the plane of the shadow, will be the straight outlines of the shadow of the solid; and lines drawn from the points of contact in that base, or in that section, on the conical surface, to the apex, will separate the illuminated from the unenlightened part.

If the shadow of the apex of a cone fell within that of its base, the outline of the shadow of the base would obviously be that of the whole solid, and all the conical surface would either receive light, though unequally, unless the axis of the cone passed through the luminary, or would be wholly in shadow if the base were nearest the luminary.

The case in the cylinder, analogous to this last respecting the cone, is presented, if a line through the luminary parallel to the axis falls within the cylindrical surface, when the whole shadow of the cylinder will be that of the base of the solid nearest the luminary.

Part of the shadow of the cylinder of the last example is intercepted by the surface of the other: to obtain the outline of this portion, it would be necessary to find the intersections with both solids of any planes passing through the luminary and its plan; for each such intersection would be cut, on the cylindrical surface receiving the shadow, by the ray of the point in which the plane cut the circular edge, *m f s q*, casting the shadow on that surface, in the shadow of that point; and a sufficient number of such points being found, the shadow can be drawn through them.

If the axes of both cylinders were parallel, this construction would be simple enough; for all planes passing through the luminary parallel to those axes would cut the cylindrical surfaces of the two solids in straight lines parallel to the axes; and as these sections could be easily drawn, the shadow of the point in the section on one surface could be immediately obtained on the other by the ray of that point. But when, as is the case with the two cylinders in the figure, the axes are not parallel, any plane passing through the luminary, and parallel to the axis of the solid casting the shadow, would cut the other cylinder in an elliptic section, and to draw this would be a troublesome operation; it is, generally, therefore only done for one such section, so as to obtain a point in the shadow on the curved surface, in the situation which will best serve as a guide for drawing it.

In the present case, the elliptic outline of the shadow of the arc *p m s q* cuts the section of the other cylinder *w x*, &c., in two points, from which the curve of the shadow on the cylinder must begin: again, for reasons already stated, this curve must touch the outermost ray of the cylinder of rays casting the shadow, and these three data would be sufficient, if one other point were found by a sectional plane, (which should be taken so as to give a point in the curve near to the letter A,) to allow of the curve being drawn tolerably correctly. The elliptic section of the cylinder by such a sectional plane must be found in the manner in which that of the first cylinder by the plane (D K, E C) was obtained (see p. 217); that is, the rhombus formed by the section of the surrounding rectangular parallelopiped must be first drawn, and then the elliptic section inscribed within it, taking care to make it touch the sides in the proper points, that is in the lines in which the planes of the faces of the parallelopiped touch the surface of the cylinder.

If either of the cylinders in fig. 4 were supposed to be formed by the rays of its circular base, the intersection of the two cylinders, as found by the construction before given (see p. 219) would be the outline of the shadow cast by the base of the one on the surface of the other cylinder; and when the position of the axes of the two solids admit of it, that construction can be employed to determine that outline.

If it were required to determine the outline of the shadow cast by the semicircle *y m s y z* (fig. 3) on the inner concave cylindrical surface, the following construction must be used. From *r′* (see fig. 2) draw any chords at pleasure, as *g f*, to cut the circular base; draw *f* S, and the ray from the point *g* will cut this line in *f′*, the shadow of *g* on the concave surface: a sufficient number of points must be found in this way, and the curve *y f′ z* drawn through them.

For it is obvious that $\ast S$, $r'gf$, and fS, all lie in one plane, in which also the ray of the point g must lie: this ray therefore will cut fs in the shadow of g as cast on the concave surface.

The curve $yf'z$, though produced by the intersection of *two* curved surfaces, is nevertheless an *ellipse*, as will be now shown. Having drawn any chord gf from r', cutting yz in a point c,[†] draw Sf, Sg, and Sc; and since these lines and the ray $\ast gf'$ all lie in one plane, a line from f' through c must cut Sg produced in some point h; and $f'h$, if drawn and produced, must cut $S\ast$ in some point: let this point be denoted by Σ.

Now yz being the chord of the tangents from r', $r'f$ must be harmonically divided in f, c, g and r'; consequently $S\ast$, Sg, Sc, and Sf are harmonical lines[‡]; therefore $f'h$ produced is harmonically divided in c and Σ. And since $f'\Sigma$, fr', both harmonically divided, have one point c in common, and ΣS is a line joining the second points of division in each from c; the lines fh, $f'g$ joining the remaining points of division in each, must meet in a point in ΣS; but $f'g$ is the ray of the point g from $\ast$, therefore fh cuts $S\Sigma$ in $\ast$[§]. Again, since $S\Sigma$ is harmonically divided (see demonstration in note), and since the foregoing demonstration applies to any and all chords drawn from r' to cut the circular base, the fourth point Σ of division of $S\ast$ will be the same, the three others remaining constant. Hence therefore the lines Σhc all lie in a plane passing through Σ and yz; and the curve of the shadow $zf'y$ is the section of the cylinder by this plane, and is therefore an ellipse.

When the chords $r'gf$ become the tangents $r'y$, $r'z$, the line $\Sigma hcf'$ become tangents to the curve of the section $yf'z$, as well as to the cylindrical surface; let the chord yz be bisected by Σc;

[†] It is not necessary to the demonstration that gf should pass through the centre of the base; nor would yz be a diameter to that base if the luminary were at a finite distance: and the construction and demonstration apply generally whatever may be the situation of the luminary.

[‡] It will be perceived, that the demonstration equally applies whether the original lines, or their projections on the plane of the picture, are considered: in the former case, $S\ast$, Sf, &c. are harmonical *parallels*, S being a vanishing point; in the latter supposition these lines will be harmonical lines. For the *original* of $r'f$ being harmonically divided, its *image* will be harmonically divided: if, as in the example, $\ast$ be at an infinite distance, yz will be a diameter, and will bisect the parallel chords gf drawn from the vanishing point r; in this case the original of gf being bisected, its image will be harmonically divided in the images of f, c, g, and in r', the vanishing point of the original chord.

[§] For if fh, for example, did not cut $f'g$ produced, in the same point $\ast$ in which $f'g$ cuts $S\Sigma$, but in some other point (p); then fhp would *not* cut $S\Sigma$ in $\ast$. Now on account of the harmonical division of fr', the lines $f'f$, $f'c$, $f'g$, and fr' will divide $S\Sigma$ harmonically in S, Σ, $\ast$ and r': again, on account of the harmonical division of $f'\Sigma$, the lines ff', fc, $f\Sigma$, and fh, will divide $S\Sigma$ harmonically in S, Σ, r', and in the other point *not* being $\ast$: that is, the same line $S\Sigma$ would be divided by two sets of harmonical lines in points, three of which produced by each set would coincide, and the two fourth points of division would not coincide, which is impossible; therefore fh must cut fg' and $S\Sigma$ in $\ast$.

through the points h and f', draw lines to p, the vanishing point of $y\,z$; these lines will be tangents to the curve of the section and the quadrilateral formed by them, and by $\Sigma\,y$, $\Sigma\,z$, will allow of the elliptic section being correctly drawn.

If the luminary be at an infinite distance, as in the example, the tangents $\Sigma\,y$, $\Sigma\,z$ will be parallel to each other, and perpendicular to $y\,z$, and therefore perpendicular to the tangents $h\,p$, $f'\,p$ which are parallel to $y\,z$; the quadrilateral formed by the tangents will be a rectangle. If the luminary be at a finite distance, the tangents $\Sigma\,y$, $\Sigma\,z$ will not be parallel, and the quadrilateral will not be rectangular.

If the axis of the cylinder be parallel to the plane of projection S, the vanishing point of its axis, one point of the harmonical division of Σ S, being in that case at an infinite distance, $*$ will bisect $r'\,\Sigma$, or $r\,*$, $*\,\Sigma$ will be equal. If, therefore, a line were drawn through s, the plan of the luminary in E C parallel to the axis of the cylinder $y\,z$, or perpendicular to E C, then the length of the support of $*$, $s\,*$, being set off from $*$ along this line, will give the point Σ at once, and the whole construction will be much simplified.

The preceding constructions and demonstration apply whether the cylinder be right or oblique, and for reasons before stated are, with obvious modifications, equally applicable to the right and oblique cone.

The shadow cast by any curved surface on a plane, or on any other surface, may be found by describing the sections of the surfaces, as made by planes passing through the luminary: for rays lying in these sectional planes, and drawn tangents to the curve of the section, on the solid casting the shadow, will cut the curve of the section on the *plane* or *surface* of the shadow, in the shadow of the tangent point. And a line drawn through all the points where the tangential rays touch the former curves, will separate the light from the shade on the surface.

A B F G, &c. (Pl. 5, fig. 5) is the *plan* of half a square abacus surmounting an annulus on the top of a cylinder; the three solids forming a rude resemblance to the capital of a column, as is seen from the *elevation* above; this elevation being the orthographic projection of the capital on a plane perpendicular to that of the plan, and passing through the axis of the cylinder and ring.

Let S C, s C be the plan and elevation of a solar ray; draw any lines as 4 1, parallel to S C; these lines will represent the intersection with the plan of planes perpendicular to that of the plan. Draw perpendiculars to A B through the points 4, 3, 2, 1, &c., in which these intersections cut the lines and circles which are the plans of the different solids composing the capital; these perpendiculars will give the points on the corresponding lines of the elevation, and by means of them the line 2 2 q 3 p . . . 4, &c. representing the section by the cutting plane[†] may be drawn on the elevation. Draw a ray parallel to s C to touch the section of the annulus in q, and to cut the section 2 2 on the cylinder; and also draw a ray through 4, the point on the lower edge of the abacus in which it is cut by the section, to cut the curve on the annulus in p: then lines drawn through the points p on the annulus will be the outline of the shadow of the abacus on the surface; and the outline of the shadow (r . . h . .) cast by the annulus on the cylinder must be drawn through the points on the cylinder as thus found, and the line drawn through the points q will separate the light and shade.

The line 4 3 drawn on the plan to touch the circle A T B, will give the point 3 on the elevation, through which the line of light and shade will pass; the line of light and shade on the cylinder is found in the same way, by drawing a ray on the plan to touch the circle D H 2, which represents the cylinder; and if the section of the annulus and abacus by the plane passing through this ray be drawn, (as is done in the figure,) the ray drawn tangential to the curve on the annulus produced by this plane will give the point where the curve r . . h . . of shadow on the cylinder falls into the shaded part of that solid.

For rays lying in the planes between T 2 and 4 3 on the plan, will not touch the cylinder, though they will still be intercepted by, and therefore illuminate the annulus; while rays lying in planes beyond 4 3 (nearer to G) will not fall on either surface[‡].

[†] The section of the annulus by the plane (4 1) will not be an ellipse, as may be seen by referring to fig. 2 in the same plate.

[‡] Of course in treating the subject geometrically, all considerations of the effect of the atmosphere, in softening the separation of light and shade on a curved surface, are omitted; as well as of the effects of reflected light, in causing the part of the surface in shade to vary in the intensity of the shadow: these, and the effects of the *foreshortening* of the curved surface as it recedes, and as the rays to the vertex form an acuter angle with that surface, thus modifying the reflected light on the shade so as to cause the part near the apparent outline of the surface to darken again, are more properly treated of in works on painting, chiaro oscuro, &c.

One effect of the last-mentioned cause, namely, the foreshortening of the curved surface, may be mentioned here; the line on which the rays lying in the plane (S C) passing through the axis would be the lightest part of the curved surfaces, because those rays would fall most directly on them in that line; but that would not appear to the eye the line of highest light, unless the plane of those rays also passed through the vertex or eye; for if this were not the case, the line of apparent high light would be that on which those rays would fall, which, being reflected in an angle equal to that of incidence, would arrive at the eye. To find this line would require complicated constructions; but for the cylinder in the example, the following will give an approximation to it. Draw C H to bisect the angle S C T; then H h drawn perpendicular to A B will be the line of light nearly: for a ray parallel to S C falling on the circle at *H*, would be reflected in a line parallel to C T, or would pass through the vertex (the projection being orthographic.)

The section by a plane of rays, passing through the angle F of the abacus, must be found, in order to obtain the shadow of the angle of the abacus as cast on the annulus at *n*; and to obtain the portion of the outline which lies between *n* and the contour of the moulding, a profile elevation on a plane passing through C T must be employed; but it is not needful to dwell further on this proceeding.

The same principle is pursued in finding the shadow, &c. of the annulus perspectively projected (Pl. 5, fig. 2), a series of sections of the solid, by planes parallel to the faces of the circumscribing parallelopiped H M, J L, these faces having been assumed as passing through the infinitely distant luminary, and perpendicular to the planes K M, I L (see p. 228, Note †), for the sake of facilitating the subsequent constructions.

Let Y X be the *intersecting line* of one such cutting plane; Y X being made equal to H I, and Y *o*, X *n* drawn to the vanishing point; the sides *l m*, *o n* of the parallelogram, circumscribing the section of the annular surface, must be deduced from a plan of the ring on the original plane.

If a plane parallel to I L, J N were to pass through the points in the diagonals of the squares circumscribing the circular central sections at I and 2, in which the circles cut those diagonals, these horizontal planes will cut the oblique sections, *l m n o*, in lines parallel to *l o*, *m n*, and the points in these parallels to *l o*, *m n*, in which the curve of the section cuts them, can be deduced from the plan of the solid with its circumscribing squares, as will be easily understood without further description.

Having thus obtained a sufficient number of sections, the tangential rays must be drawn to each, cutting the intersection Y *o* of the cutting planes with the plane of the shadow, in the shadow of the tangent point, and thus points in the outline of the shadow, and in the line separating the light and shade on the annular surface, may be obtained, as was explained of fig. 5.

The planes M H, L J touch the solid in the points 2; and the line H P will be a tangent to the shadow at *b*, the shadow of 2, 2 *b* being the ray of 2. And in like manner the planes parallel to (H M), which touch the inner side of the annular surface, will cut the plane of the shadow in lines, tangents to the shadow cast by the inner surface.

The series of rays from a luminous point, which are tangents to the surface of a sphere, form a right cylinder, or a right cone, according as the luminary is at an infinite or finite distance: these rays will touch the sphere in a circle, the plane of which will be perpendicular to the line drawn from the luminary through the centre of the solid: and if this circle be found on the sphere, the shadow of it on any surface will be that of the sphere.

Let the luminary * (Pl. 8, fig. 3) be given at any finite distance, and let Hy be the vanishing line of the plane of the shadow: find l the *plan* of * (see p. 271, and fig. 4), and f the *plan* of the centre F of the sphere, on the plane of the shadow. A line drawn through l and f will be cut by the ray * F in f' the shadow of the centre of the sphere. (12. p. 269.) Produce lf to y its vanishing point, and draw y S parallel to * l, or perpendicular to Hy, then S will be the vanishing point of the ray * F (10, p. 269); draw S C through the centre of the picture, and draw C V′ perpendicular to S C, and equal to the distance of the vertex (C V): draw the radial V′S, and V′R perpendicular to it, to cut S C produced in R: through R and S draw lines perpendicular to S R for two vanishing lines of planes perpendicular to each other; and R and S will be the centres of these vanishing lines.

Draw a diameter of the sphere (equal to de, see p. 262) parallel to these vanishing lines last found; describe the projection of a square $rstu$ lying in the plane passing through the centre of the sphere, and having the vanishing line through S for its vanishing line[†], and having the side of the square equal to the diameter de of the sphere, and its centre in common with that of the solid. In this square inscribe the ellipse which represents the circle produced by the section of the sphere by the plane of the square.

Draw from * two tangents to the elliptic projection just found, and draw the chord joining the points of contact parallel to rs, or to the vanishing line, this chord cutting * S in F. Then describe the projection W Y X Z of a square lying in a plane passing through F, having the vanishing line through R for its vanishing line, and having its sides equal to the chord of the tangents just found.

The ellipse inscribed in W Y X Z will be the projection of the circle[‡] on the spherical surface, in which the cone of rays from* will touch the sphere; and the shadow of this circle on the given plane will be the shadow of the solid as required.

For S, the centre of the vanishing line, being the vanishing point of the ray, * F, of the centre of the sphere, being also made that of the sides rt, su of the square; and since the diagonals

---

[†] It is almost needless to remark that the radial S V′, set off from S along the vanishing line each way, will give the vanishing points of the diagonals of the square ru, st, by means of which the projection is directly found (42. p. 149.) So also in the subsequent step, the radial R V must be set along the vanishing line from R each way for the vanishing points of the diagonals of W Y X Z.

[‡] The elliptic projection of this circle ought, if the constructions have been rightly made, to touch the outline of the sphere.

Google

of the square pass through F, the plane of that square must pass through the luminary, and the sides being made perspectively equal to *d e*, the diameter of the sphere, the sides *r s*, *t u* are parallel to the plane of the picture: consequently, the chord of the tangents to the inscribed circle from *, will also be parallel to the plane of the picture, and those tangents will be two of the rays forming the cone of rays above mentioned.

Hence, the circle described on that chord of the tangents for a diameter, and lying in a plane perpendicular to * *F*, will be the circle forming the base of the right cone of rays: now by its vanishing line through R, and by its diagonals passing through *F*, the square W Y X Z lies in such a plane.

To draw the shadow of the circle just found, it will be best to describe that of the circumscribing square W X Y Z. Draw R H perpendicular to *y* H; then H will be the vanishing point of the plans of Y Z, W X, &c. (9. p. 269): draw the lines R * and *l* H, cutting each other in *a*; then the shadows of Y Z, W X, &c., will pass through *a* (16. p. 270).

Through * draw a line parallel to the vanishing line through R; and through *l* draw a line parallel to *y* H, meeting the former line in a point *b*,† through which point the shadows of W Y, X Z will pass (16. p. 270). Through *f′* and *b* draw a line which will be cut by the rays of the extremities of the diameter of the circle through *F* in the shadows of those extremities; through *f′* and *a* draw a line, and find the shadows in it of the extremities of the diameter through *F*, which is at right angles to the former. Then lines to *b* and *a*, through the shadows of the extremities of these diameters, as just found, will complete the projection of the shadow of the square W X Y Z; and an ellipse inscribed in this shadow touching it in the proper points will be that of the sphere as sought. Additional guides to drawing this shadow will be furnished by rays drawn tangents to the outline of the sphere; for these rays will obviously also be tangents to the shadow.

If the plane of the shadow were parallel to the ray * F, the centre of the sphere would have no shadow, and that of the solid would be an hyperbola. And the same shadow would be a parabola, if the plane of the shadow were parallel to a side of the cone of rays.

If the plane of the shadow touched the sphere, the point of contact would be one focus of the curve of the shadow; and the common intersection of the plane of the shadow, and of that of the small circle, constituting the base of the cone of rays, would be the directrix to the curve. (Geom. App. Prop. 21.)

---

† *b* is far out of the plate, but the lines * *b*, *l b*, indicate its direction.

The shadow of the edge of a hollow hemisphere, as cast on the opposite concave surface, may be found by the same principles as that of the edge of the cylinder was (see p. 284): this shadow will be an *ellipse*, though produced by the intersection of two curved surfaces: the demonstration of this being closely analogous to that of the corresponding property in the cylinder, will not be required by the learner who has made himself master of the one referred to.

The shadows of solids having surfaces of any species of curvature may be found, if required, by an extension of the principles explained above for finding those of the *cylinder*, the *annulus*, and the *sphere*; but it is not necessary in an elementary work, as this is, to enter into the constructions required in these more complicated cases.

NOTES AND ILLUSTRATIONS

TO THE

PLANE GEOMETRY.

NOTES AND ILLUSTRATIONS

TO THE

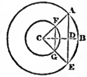
PLANE GEOMETRY.

Note A, p. 7.

The distance between the points of a common pair of compasses, is obviously the *chord* of the angle formed by the two legs: in the proportional compasses, the angles formed by the legs on each side of the centre, or joint, are equal, they being vertical angles (Geom. I. Prop. 3); and the distances between the opposite pairs of points is proportional to the distance of those points from the centre of the joint, or proportional to the radii of the circles measuring the equal angles (Geom. III. Prop. 33, Cor. 1): if therefore the joint is set so that these distances shall be in any proposed ratio, the distances between the opposite pairs of points will be in the same ratio.

Note B, p. 24.

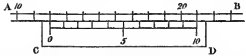
The *sine* A D, of any angle A C B, is equal to half the *chord* A E, of double the angle, or half the chord of the angle A C E. Now the chord F G of the same double angle, as subtended in a circle of the radius C F, equal to half C A, is equal to the sine A D: for F G is equal to half of A E (Geom. III. Prop. 33, Cor. 1). If therefore the *sine* A D of half the proposed angle be taken from a table, calculated to the radius unity C A, this sine may be set off as the *chord* of the required angle A C E on a circle of half unity, or C F for radius; as is directed to be done in the text.



Note C, p. 28.

The mode of subdividing small distances, now universally used in mathematical and philosophical instruments, is by means of what is called, from the name of its inventor, a *Vernier*.

If n number of divisions on any graduated scale A B be divided into



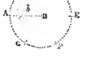
$n+1$ number, on another moveable scale C D, which is the vernier, each division on the latter, will be less than a division on the original scale by $\frac{1}{n}$ part of a division, as will be understood from the figure.

If C D were moved along A B till the first division of the vernier coincided with the thirteenth division on the scale, the O division of the former would be $\frac{1}{10}$th of a division in advance of the twelfth; and if C D were moved till its second division coincided with the fourteenth of the scale, then the O division would be $\frac{2}{10}$ths in advance of the twelfth; and so on, till, if C D had moved so that the ninth division of the vernier was opposite the twenty-first of the scale, the zero of the vernier would show 12·9 on the scale. Hence, therefore, when by the action of the instrument, of whatever kind it may be, to which the vernier is attached, the zero of this vernier falls between two divisions of the graduated *limb*, the fraction of a division by which it is in advance of the last is known by observing the number of divisions of the vernier at which the first coincidence takes place, then that number indicates the fraction sought. Thus if it was seen that the seventh division from zero coincided with the nineteenth division of the scale, zero would indicate 12·7.

On astronomical instruments of the smaller kinds, as sextants, &c., the *limb*, or graduated edge, is usually divided into degrees, each of which is subdivided into three parts for twenty minutes: then nineteen of such subdivisions are divided on the vernier into twenty, so that each subdivision is by this means divided by inspection into single minutes.

NOTE D, p. 30.

N.B.—As the steps in the following demonstration will have to be frequently referred to in subsequent notes, containing the demonstrations of the constructions by the compasses alone, in the solution of several problems in the text, more are given here than are absolutely necessary for the demonstration of this particular case. The reader must supply several straight lines in the figures, which are required for those demonstrations, but which were not shown in the diagrams, to avoid complicating them: and he must understand that the straight lines which are drawn in the diagrams, belonging to solutions by the compasses alone, are inserted only for the sake of demonstration, none being required or admitted in the construction.



1. If A B, the radius, be taken as unity, A E, the diameter, will be equal 2; and since the triangle A D E is right-angled (Geom. III. Pr. 15, Cor. 1): $AD^2 = AE^2 - DE^2 = AE^2 - AB^2 = 4 - 1 = 3$; therefore $AD = \sqrt{3}$.

2. Again, A B a being a right-angled triangle, $Ba^2 = Aa^2 - AB^2 =$ (by construction) $AD^2 - AB^2 = 3 - 1 = 2$; and $Ba = \sqrt{2}$.

3. A D F, B D E being equilateral triangles, A E perpendicular to D F bisects it, and D F perpendicular to B E bisects it in a point p: and let a B cut the circumference in a point q: join pq.

It is clear from the construction, that the point b must fall in the diameter A E; and D F being equal to $AD = \sqrt{3}$, $Dp = \frac{\sqrt{3}}{2}$, Dp^2

$= \frac{3}{4}$; and $b\,p\,\mathrm{D}$ being a right-angled triangle, and $b\,\mathrm{D}$ by construction being equal to $\mathrm{B}\,a$: $b\,p^2 = b\,\mathrm{D}^2 - \mathrm{D}\,p^2 = 2 - \frac{3}{4} = \frac{8}{4} - \frac{3}{4} = \frac{5}{4}$.

Again, because $p\,\mathrm{B}\,q$ is a right-angled triangle, $p\,q^2 = \mathrm{B}\,q^2 + \mathrm{B}\,p^2 = 1 + \frac{1}{4} = \frac{5}{4}$. Therefore, $b\,p = p\,q$.

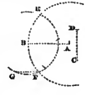
4. $\mathrm{E}\,p$ being equal to $p\,\mathrm{B}$, $\mathrm{E}\,b = p\,b + p\,\mathrm{B}$: and $\mathrm{B}\,b = p\,b - p\,\mathrm{B}$. Therefore, $\mathrm{E}\,b \cdot \mathrm{B}\,b = p\,b^2 - p\,\mathrm{B}^2 = p\,q^2 - p\,\mathrm{B}^2 = \mathrm{B}\,q^2 = \mathrm{A}\,\mathrm{B}^2$; that is, $\mathrm{E}\,b \cdot \mathrm{B}\,b = \mathrm{B}\,\mathrm{E}^2$, therefore $\mathrm{E}\,b$ is medially divided in B.

5. Again, since $\mathrm{E}\,b \cdot \mathrm{B}\,b = (\mathrm{E}\,\mathrm{B} + \mathrm{B}\,b)\,\mathrm{B}\,b = \mathrm{A}\,\mathrm{B}^2 = \mathrm{A}\,\mathrm{B} \cdot \mathrm{B}\,b + \mathrm{B}\,b^2 = \mathrm{A}\,\mathrm{B}\,(\mathrm{A}\,\mathrm{B} - \mathrm{A}\,b) + \mathrm{B}\,b^2 = \mathrm{A}\,\mathrm{B}^2 - \mathrm{A}\,\mathrm{B} \cdot \mathrm{A}\,b + \mathrm{B}\,b^2$. We have $\mathrm{A}\,\mathrm{B}^2 = \mathrm{A}\,\mathrm{B}^2 - \mathrm{A}\,\mathrm{B} \cdot \mathrm{A}\,b + \mathrm{B}\,b^2$, and by adding and subtracting, $\mathrm{A}\,\mathrm{B} \cdot \mathrm{A}\,b = \mathrm{B}\,b^2$; that is, the radius A B is also medially divided in b. (See Schol. to Prop. 59, Geom. II.)

Note E, p. 31.

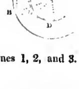
It is obvious by the construction, that A B is perpendicular to E F; and since E F G is a semicircle, E F G is a right angle, or G F is perpendicular to E F: therefore, G F, B A are parallel.

And since the triangles A B F, B F G are both isosceles, and have the angles B F G, F B A equal, the triangles are similar. Therefore, $\mathrm{A}\,\mathrm{B} : \mathrm{B}\,\mathrm{F}\,(\mathrm{C}\,\mathrm{D}) :: \mathrm{B}\,\mathrm{F}\,(\mathrm{C}\,\mathrm{D}) : \mathrm{G}\,\mathrm{F}$; that is, $\mathrm{A}\,\mathrm{B} \cdot \mathrm{G}\,\mathrm{F} = \mathrm{C}\,\mathrm{D}^2$. Q. E. D.



Note F, p. 31.

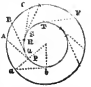
Since $\mathrm{B}\,\mathrm{D} = \mathrm{C}\,\mathrm{E}$ by construction, and $\mathrm{O}\,\mathrm{C} = \mathrm{O}\,\mathrm{B}$ and $\mathrm{O}\,\mathrm{E} = \mathrm{O}\,\mathrm{D}$, the triangle O B D has its sides respectively equal to those of the triangle O C E; these triangles are therefore similar, and the angle B O D is equal to the angle C O E: from (or to) each of these angles take away (or add) the common angle B O E; then C O B will be equal to D O E; and as the triangles D O E, B O C are both isosceles, and have the angles at their vertices equal, they are similar.



Therefore $\mathrm{C}\,\mathrm{O} : \mathrm{O}\,\mathrm{E} :: \mathrm{C}\,\mathrm{B} : \mathrm{E}\,\mathrm{D}$; that is, by construction, E D is a fourth proportional to the given lines 1, 2, and 3. Q. E. D.

Note G, p. 34.

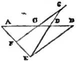
That the chords on the inner circle, as found by the construction, will be in the same proportion, in succession, to each other, that the given chords set off on the outer circle are, is proved by the demonstration in Note F. And since the similar arcs of circles are as the radii (Geom. III. Prop. 33, Cor. 1), the two sets of chords are in the given common ratio of $\mathrm{P}\,\mathrm{Q} : \mathrm{Q}\,\mathrm{R}$.



And since similar polygons are as the squares of their homologous sides (Geom. II. Prop. 43, Cor.), by making the radii of the circles as the square roots of the terms of the proposed ratio, the areas of the polygons are to be in; the chords, or *sides* of the polygons, will be respectively as those square roots, therefore the *areas* of the figures will be in the given ratio.

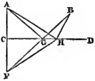
Note H, p. 37.

By similar triangles A F C, A E B; A C : C F (C G) :: A B : B E; but from the similar triangles D C G, D E B; C D : C G :: D B : B E; therefore A B : A C :: D B : C D. Q. E. D.



Note I, p. 39.

A C being made equal to C F, and the angles at C being right angles, the two triangles A C G, F C G are equal and similar, therefore the angles A G C, F G C are equal, but C G F and B G H are equal; therefore, A G C, B G H are equal. Q. E. D.



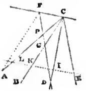
The two lines A G, B G, drawn from two points as A, B, to make equal angles with a given right line C D at a point G in it, are together less than any other two lines, as A H, B H, drawn from the same points, and meeting in any other point than G of the given line.

Having made C F as before equal to A C in A F drawn perpendicular to C D: from H draw H F. Then, in the same manner that F G was shown to be equal to A G, F H may be proved equal to A H: therefore A G + G B = F B, are together less than A H + H B = F H + H B. (Geom. I. Prop. 10.) This proposition is alluded to in a subsequent note.

If C D were a reflecting surface, the point A would be seen by an eye at B in the point G, from the known law of equal reflection and incidence of rays of light. Hence this proposition affords a mode of measuring the height of a building, or other object which is inaccessible: for if a basin of water be set on the ground, as at G, and the person move back till he sees the top of the object A in the water, he obtains the following data for solving the problem: 1st, the height of his eye above C D, or the length of a perpendicular to C D from B; 2nd, the distances of the point G from C, and from the foot *p*, of the perpendicular last mentioned. Then the right-angled triangles A C G, B *p* G being similar, as G *p* : *p* B :: G C : C A, the altitude sought.

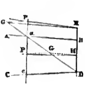
Note J, p. 40.

Draw F A parallel to C B to meet C A in A; then because P G : P F :: P F : P D by construction, and the triangle P F A is similar to the triangle P G C; as P G : P D :: tri. P G C : tri. P F A (Euc. 6, Prop. 20, Cor. 2). But as P G : P D :: tri. P G C : tri. P D C (Geom. II. Prop. 39); therefore tri. : P F A = tri. : P D C. Add to both of these equal triangles, the triangle P F C; then F A C will be equal to F D C; and as these two triangles stand on the same base F C, the line A D joining their vertices must be parallel to F C.



And because of the parallels F D, C E and F A, C B; A D will have segments equal to F C, and therefore equal to one another intercepted by A C, B C and by D C, E C: consequently the segments of any other line parallel to A D or to C F, will also have equal segments as L K, I E intercepted between the same lines. Q. E. D.

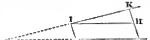
NOTE K, p. 41.



P G being parallel to c D, and G H being parallel to a B by construction, a G : G D :: a P : P c; and as a G : G D :: B H : H D, therefore a P : P c :: B H : H D: and since proportional segments of the two parallels $a\,c$, B D are intercepted by A B, P H, C D, these three lines must pass through the same point. (Geom. II. Prop. 30.) Q. E. D.

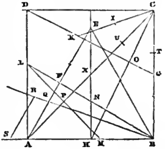
NOTE L, p. 42.

Since by the construction of the instrument, I H, I E always move parallel to E F, K H; the angles I E P, K H I are always equal; and the angle at P, in K I produced, is always equal to K I H. Therefore as K H : I E :: H I (F E) : E P, that is by adding K H + I E or K F : I E :: F E + E P or F P : E P; and since K F, I E, F E remain constant, the fourth term, or E P, must always be the same. That is, K I, if produced, would always pass through the same point P in F E produced. Q. E. D.



NOTE M, p. 49.

1. $AH = HX = \frac{1}{2}$; $AH^2 + HX^2 = AX^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$, or $AX = \sqrt{\frac{1}{2}}$. A F B, B A L being right-angled triangles, and having the angle at B common to both, $AB^2 = AF^2 + FB^2$, or $1 = \frac{1}{4} + \frac{3}{4}$ or $FB^2 = \frac{3}{4}$: and as $FB^2 : AB^2 :: AB^2 : BL^2$, or as $\frac{3}{4} : 1 :: 1 : \frac{4}{3}$; therefore $BL^2 = \frac{4}{3}$, and $BN^2 = \frac{1}{3}$ or $BN\ \sqrt{\frac{1}{3}}$.



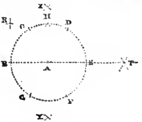
The triangles D C G, C B H being equal and similar; the angle B C H being taken from the right angles, leave D C O = B H C = D G C: therefore G O C = C B H = a right angle; therefore $DG^2 : CG^2 :: CG^2 : OG^2$, or $\frac{5}{4} : \frac{1}{4} :: \frac{1}{4} : \frac{1}{20} = OG^2$ and $CG^2 - OG^2 = CO^2$, or $\frac{1}{4} - \frac{1}{20} = \frac{4}{20} = \frac{1}{5}$ and $CO = \sqrt{\frac{1}{5}}$.

The equilateral triangle A B E having its base, A E, bisected; the angle A B F = 30°, then A L B = 60°: and since the angles B A N, A B N, are each equal 30°, the angle L A N = 60°; therefore L A = L N = B N = $\sqrt{\frac{1}{3}}$, and $LA^2 + AM^2 = \frac{1}{3} + \frac{1}{3} = \frac{2}{3} = LM^2$ and L M = $\sqrt{\frac{2}{3}}$, and $AP = PM = \frac{LM}{2} = \sqrt{\frac{2}{12}} = \sqrt{\frac{1}{6}}$.

By comparing the triangle A Q B with the triangle B C T (see fig. in second part of construction and demonstration), they will be found to be similar, and A B Q, S B R being also similar, as $QB^2 : QA^2 :: RB^2 : RS^2$, that is, 7 : 1 :: 1 : $\frac{1}{7}$ and R S = $\sqrt{\frac{1}{7}}$.

Since C X = $\sqrt{\frac{1}{2}}$; $CU = \frac{CX}{2} = \sqrt{\frac{1}{8}}$. And $CT = \frac{AB}{3} = \sqrt{\frac{1}{9}}$. And lastly, since C O = $\sqrt{\frac{1}{5}}$, $DO^2 = DC^2 - CO^2 = \frac{5}{5} - \frac{1}{5} = \frac{4}{5}$; and D O = $\sqrt{\frac{4}{5}}$ and D K or $KO = \frac{DO}{2} = \sqrt{\frac{1}{5}}$, $KC^2 = KO^2 + OC^2 = \frac{2}{5}$, K C = $\sqrt{\frac{2}{5}}$ and $CI = \frac{KC}{2} = \sqrt{\frac{1}{10}}$.

2. That A X = $\sqrt{2}$; and B D = $\sqrt{3}$ (see Note D): that B E, the diameter, = $\sqrt{4}$ is obvious. And since by construction A B R is a right angle, B R being parallel to A H, $BE^2 + BR^2 (AB^2) = ER^2 = 4 + 1 = 5$ or E R = $\sqrt{5}$.



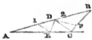
It is obvious that B, A, E, T are in a line, and that the triangles B D E, T D A are equal and similar; therefore E T = B A and B T = 3; $TX^2 = AT^2 + AX^2 = 4 + 2 = 6$ and T X = $\sqrt{6}$.

Let C G be supposed drawn to bisect B A in a point *p*. Then $CT^2 = AC^2 + AT^2 + 2\,pA \cdot AT = BC^2 + AT^2 + BA \cdot AT$ (Geom. I. Prop. 37). But $AT^2 + BA \cdot AT = BT \cdot AT$ (Geom. 1. Pr. 31). Therefore $CT^2 = BC^2 + BT \cdot AT = 1 + 3 \cdot 2 = 7$ and C T = $\sqrt{7}$.

Since A X = $\sqrt{2}$; X Y or 2 A X = $\sqrt{8}$: and since B T = 3 B A; B T = $\sqrt{9}$. And lastly, $TR^2 = BT^2 + BR^2 = 9 + 1 = 10$ or T R = $\sqrt{10}$.

It should have been remarked that, of course, only that part of the constructions which the student requires need be made.

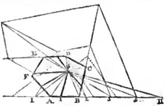
Note N, p. 52.



The triangles A C 1, 1 C 2, 2 C B are equal: and since 1 E D, 1 E C are also equal, if each of these be added to A 1 E, the whole A D E will be equal to the triangle A C 1. In the same way B D F can be shown to be equal to B C 2. Therefore the remaining triangle E D F must be equal to 1 C 2. Q. E. D.

Note O, p. 55.

To explain more clearly the principles of the constructions in this problem, the demonstration of the last case is here given at length



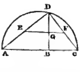
The triangle G I H, which is made equal to the given figure, is divided into equal triangles I G 1, 1 G 2, 2 G 3, &c. Now because of the parallel to A G through I, a triangle on the same base A G, and having its vertex in the side A F of the given figure, is found equal to the triangle A G I; this triangle equal to A G I, being added to the common one A G 1, makes the quadrilateral, comprised within the given figure, equal to one of the triangles I G 1, into which the triangle equivalent to the given figure was divided.

In the same way the quadrilateral formed by adding a triangle equal to G B 2, to the common triangle G 1 B, may be shown to be equal to another aliquot part of the equivalent triangle I G H. The next step is to get a figure, included within the periphery of the given figure, equal to the third aliquot triangle 2 G 3; this is accomplished by making the whole space from G 1 equal to the *two* triangles 1 G 3: of these, 1 G B is already within the periphery. Draw a parallel to G B through 3, to cut B C produced in a point (p); then the triangle G B p is equal to G B 3. Of this triangle G B p, a portion G B C falls within the periphery, and there only remains an equivalent to the triangle G C p to be obtained, that shall also fall within the periphery. To obtain this equivalent, a parallel to C G is drawn through p to cut C D produced in q: then the triangle G C q is equal to G C p. Of G C q, a portion G C D falls within the periphery, and there remains G D q to obtain an equivalent for: this is found by drawing a parallel to G D through q to cut D E in r; then G D r being equal to G D q, a portion of the given figure G 1 B C D r G is cut off equal to the two triangles 1 G 2 and 2 G 3. But a part of this space has been shown to be equal to the triangle 1 G 2; therefore the remainder is equal to 2 G 3, as was required.

By pursuing precisely the same plan, the space G 1 B C D E F G is cut off equal to the *three* triangles 1 G 2, 2 G 3, 3 G 4; and as G 1 B C D r G has been already proved equal to 1 G 2 and 2 G 3 together, the remaining portion is equal to the third triangle 3 G 4. And as the whole figure is equal to the five equal triangles into which I G H was divided, and as spaces have been cut off by lines from G equal respectively to four of the five triangles, the remainder of the figure must be equal to the remaining triangle. Q. E. D.

Note P, p. 56.

E F being parallel to A C, E G : G F :: A B : B C :: area of given rect. to area of required rectangle, by construction. And from the



semicircle on A C, E D F is a right-angled triangle, and D G being a perpendicular to the hypothenuse, as E D : D F :: E G : G D, and as E D : D F :: G D : G F (Geom. II. Prop. 34); by multiplying we have $ED^2 : DF^2$ :: E G·G D : G F·G D :: E G : G F, that is, the areas of the rectangles are in the ratio of the squares of E D, D F; E D, D F are therefore the homologous sides of similar rectangles, the areas of which are in the proposed ratio. Q. E. D.

Note Q, p. 59.

In the common pentagraph, the parallelism of the rules being always preserved, the triangles formed by them, and having A C, B C for their bases, are always similar. And as the lengths of the rules, or the antecedents of the proportion, are constant, the *ratio* of A C : B C remains the same, and A, B, C must always be in a line.

In the eidograph, the triangles C B G, C A H are always similar, because the arms always move parallel to each other; and as the antecedents, or the two sides G B, B C, are always in the same constant ratio to A H, A C, the consequents C G, C H are always in the same ratio.

Note R, pp. 61—67.

Prob. 35. B E is a diameter, and A *a* is perpendicular to it, by the construction A *a* is equal $\sqrt{2}$ (see Note D); therefore $EC^2 + BC^2 = 2Aa^2 = 4 = EB^2$: consequently E C B is a right angle, or C is in the circumference of the semicircle on E B, and lies in the perpendicular A *a*; B A, A C therefore are two sides of a square. The rest of the construction requires no comment.

Fig. 1.

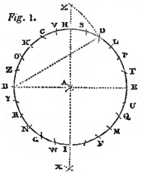
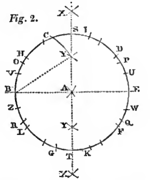


Fig. 2.



In Prob. 36, the division of the circumference into six, three, and two parts is obvious. (Geom. III. Prop. 28.) Its division into four parts is proved by the first part of this note.

On A X let a semicircle be supposed described to cut the original circle in some point *k*: join A *k*, *k* X. Then the angle A *k* X being a right angle; $AX^2 - Ak^2 = kX^2$; or (see Note D) 2 − 1 = 1 : that is, *k* X is equal to radius, and *k* coincides with K. And since the right-angled triangle A K X is also isosceles, each angle A X K, K A X is equal to 45°; therefore K A X is half H A B; consequently, the quadrant H B is bisected in K. The same proof applies to the other three points bisecting the other three quadrants.

The division into twelve parts is equally evident with that into six, being effected by the bisection of each arc of the hexagon, the arc C D being bisected in H by the previous construction.

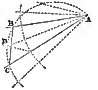
Again, the arcs K B, K H each being 45°, and K Y, K S being made arcs of 60° by the construction, the differences, or the arcs H S, B Y, are each 15° of $\frac{1}{24}$ part of the circumference, and the same proof applies to the other points.

It has been proved (Note D) that the point Y (fig. 2) divides the radius A S medially. A Y will therefore divide the circumference into 10 parts, and since $B\,Y^2 = A\,B^2 + A\,Y^2$, B Y is the side of a regular pentagon. (Geom. III. Prop. 28.)

Lastly, the arc B C, or the arc of the pentagon, being 72°, and B S the quadrant being 90°, C S is equal to 18°, and is therefore the arc of $\frac{1}{20}$th of the circumference; the arcs B V, E U, &c., are equal to C S.

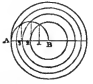
Note S, p. 69.

Since the angles B A D, D A C, &c., are equal by construction, they must stand on equal arcs of the circle passing through A B and C. (Geom. III. Prop. 14, Cor. 2.) And the chord B C being bisected by the perpendicular, that perpendicular must bisect the arc which B C subtends; the point D, therefore, must be in the circle, and B D, D C are the equal chords subtending the equal arcs on which B A D, D A C stand. The reason of the rest of the construction is obvious from what has been proved of these two angles.



Note T, p. 74.

1. From the semicircle on A B, we have $A\,1 \cdot 1\,B =$ square of ordinate at the point 1. $A\,2 \cdot 2\,B =$ square of ord. at 2, &c., &c. But the ord.^2 at $1 + 1\,B^2 = \text{rad.}^2$ of inner circle, and ord.^2 at $2 + 2\,B^2 = \text{rad.}^2$ of second circle, &c., &c.: that is, $A\,1 \cdot 1\,B + 1\,B^2 = \text{rad.}^2\,(1)$ and $A\,2 \cdot 2\,B + 2\,B^2 = \text{rad.}^2\,(2)$. &c., or $(A\,1 + 1\,B)\,1\,B = \text{rad.}^2\,(1)$, and $(A\,2 + 2\,B)\,2\,B = \text{rad.}^2\,(2)$ &c., &c. Or, since $A\,1 + 1\,B$, $A\,2 + 2\,B$, &c. are all equal to A B; $A\,B \cdot 1\,B = \text{rad.}^2\,(1)$ $A\,B \cdot 2\,B = \text{rad.}^2\,(2)$ $A\,B \cdot 3\,B = \text{rad.}^2\,(3)$ &c.



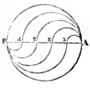
And since the areas of the circles are as the squares of their radii, the areas of the successive circles are in the ratio of $A\,B \cdot 1\,B$, $A\,B \cdot 2\,B$, $A\,B \cdot 3\,B$, &c., or in the ratio of $1\,B : 2\,B : 3\,B$, &c., or as $1 : 2 : 3$, &c.: that is, the areas of the whole circles increase by equal quantities;

and if the interior circles be taken from each in succession, the *rings* left are equal in area. Q. E. D.

2 If the area of the semicircle on A 1, one division of the diameter A B, be taken as unity; the areas of the semicircles on two, three, four, &c., divisions, will be 4, 9, 16, 25, &c. (Geom. III. Prop. 33.) Let these semicircles on one side of A B be expressed by a, b, c, d, &c., and the equal semicircles beginning from the other end of the same line and on the other side of it, by A, B, C, D, &c.: that is, $A = a$, $B = b$, &c., and let x be the area of the semicircle on A B. Then the figure which remains, when the semicircle on A 4 is taken from the semicircle on A B, is equal to $x - d = 25 - 16 = 9$. If the semicircle on $B\,4 = 1$ be added to this, the whole compound figure thus made up is $= 10$, which call P.

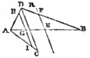
Again, if from x we take the semicircle on A 3, and add to the remainder the semicircle on B 3, we have $x - c + B = 25 - 9 + 4 = 20$, and if we take P away we have $20 - 10 = 10 = Q = P$.

In the same way $x - b + C = 25 - 4 + 9 = 30$, and by subtracting $(P + Q)$ we have $30 - 20 = 10 = R = Q = P$, &c., and so on of any other number of divisions.



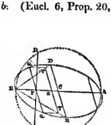
Note AA, p. 82.

Let A E, the semi-transverse axis, be denoted by a; and let the semi-conjugate be denoted by b: and let the point in which A B, C D intersect, be denoted by c.



From the similar triangles B E F, B c D; A G H, A c D; we have $AG : GH :: Ac : cD$ and $BE : EF :: Bc : cD$; and by multiplying $AG \cdot BE : GH \cdot EF :: Ac \cdot Bc : cD^2$. But from the ellipse, as $a^2 : b^2 :: Ac \cdot Bc : cD^2$; therefore, $a^2 : b^2 :: AG \cdot BE : GH \cdot EF :: BE : GH$ (because $AG = EF$ by construction) $:: a : GH$, or $a : GH :: a^2 : b^2$; consequently, G H is a third proportional to a and b. (Eucl. 6, Prop. 20, Cor. 2.) Q. E. D.

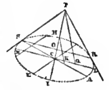
Note BB, p. 84.



On account of the circle on B A, $BP : PR :: PR : PA$, or $BP \cdot PA = PR^2 = BS^2$ by construction; and by the similar triangles B S T, B C D, we have $BS^2 : ST^2\ (PQ^2) :: BC^2 : CD^2$. And as $BP \cdot PA : PQ^2 :: BC^2$: the square of the semi-conjugate (Geom. App. Prop. 19). Then since $BS^2 = BP \cdot PA$ and $ST^2 = PQ^2$ by construction, CD^2 must be equal to the square of the semi-conjugate as required. Q. E. D.

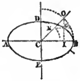
Note CC, p. 84.

N.B.—CQ, *not* LQ, *is to be made equal to* CH.



From the similar triangles CKH, CQP, $CK : CH :: CQ : CP$; therefore, $CK \cdot CP = CH \cdot CQ$, or $CO \cdot CP = CH^2$, because $CO = CK$ and $CQ = CH$ by construction. CH is therefore a mean proportional between CO and CP; and since $IO + OH$ is bisected in the point C, PI is harmonically divided in O and H (Geom. II. Prop. 46). The ordinate SR is therefore the chord of the tangents from the point P. (Geom. App. Prop. 18.) Q. E. D.

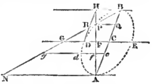
Note DD, p. 85.



KG being parallel to CI, as $CO^2 : CK^2 :: IO^2 : IG^2$ (Geom. II. Prop. 29); but since $IO^2 = CO^2 - CI^2 = (CO + CI) \times (CO - CI) = (AC + CI)\,(AC - CI) = AI \cdot IB$. And since $CO^2 = AC^2$ and $CK^2 = CD^2$, the above proportion becomes as $AC^2 : CD^2 :: AI \cdot IB : IG^2$, therefore G is a point in the ellipse. (Geom. App. Prop. 19.) Q. E. D.

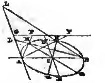
In the construction I, p. 97, let two parallel lines perpendicular to AB be supposed drawn through D and P; then since CD, DE are always equal, the perpendicular Dd through D will bisect the angle CDE. Let the perpendicular through P cut AB in p, and CD produced in q; then the angle $EPp = DPq = CDd = Cqp$, or $DPq = DqP$: consequently, $Dq = DP$; and as DP is constant, Dq is so: Cq therefore corresponds to CO (see fig. above), and when CD, DE are close together, then $CP = CD - DP$ is analogous to CD, the point P on the side answering to G as found by the geometrical construction. The semi-major axis of the ellipse as described by means of the jointed rule will be equal to $CD + DP$, when the point P comes to B, and the semi-minor axis will be equal to $CD - DP$.

Note EE, p. 85.



1. *Lemma.* If from the vertices of any diameter of an ellipse AB, lines be drawn through a point P in the curve, to cut the tangents at B and A in points H and N: then any ordinate to AB, as cd, cutting the lines AH, BN in f and g, and the curve in d, will be divided in those points in continued proportion: that is, $cf : cd :: cd : cg$.

Draw ED, the conjugate diameter to AB, and draw the ordinate PQ, which will be parallel to the conjugate and to the tangents. Then by similar triangles $AB : BH$



B A : A G :: BA^2 : DE^2. (Eucl. 6, Prop. 20, Cor. 2.)

Let an ordinate, M P, to A B be drawn from M, and therefore parallel to D E and A L; then by the similar triangles G A F, A P M, and B A L, B P M, we have G F : A G :: P A : P M, and B A : A L (G F) :: B P : P M; and by multiplying G F·B A : A G·G F :: P A·B P : PM^2, or B A : A G :: B P · P A : PM^2, or BA^2 : DE^2 :: B P·P A : P M; M, therefore, is a point in the curve. Q. E. D.

Note I I, p. 92.

A point in the circumference of the inner circle will describe a right line, which will be a diameter to the larger circle, (Les. G. C. L. Epicy.) and the opposite end of the diameter of the lesser circle passing through the first point will describe another diameter to the outer circle, at right angles to the former, these two lines being, theoretically, epicycloids. And a point P in the diameter of the inner circle must describe an ellipse, because the two vertices of that diameter, which are always at a constant distance from each other, move in two lines at right angles to each other. See following note.

Note J J, pp. 92, 95.

For the theorem, of which the trammel and the two elliptographs are a mechanical application, see Les. G. C. L.

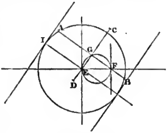
In Mr. Farey's elliptograph, the rectangular frame causes the two centres of the rings, when these are excentric, to move in two right lines perpendicular to each other; and as the centres of the circles and the point of the pencil are always in a line, and always preserve the same distances respectively from the construction of the instrument, the conditions of the theorem are fulfilled, and the pencil must describe the curve.

In Mr. Clement's, the centres are kept at the same distance by the screws when set, and the triangular bars move in directions perpendicular to each other, carrying the centres in lines at right angles; thus fulfilling the same conditions as the former instrument.

The contrivance called the *excentric chuck*, by means of which ellipses are turned in a common lathe, is also a mechanical application of the same theorem, though not so obvious as in the last mentioned instruments: and deserves to be explained here, because the large elliptograph employed by the late Mr. Lowry, the engraver, is constructed on the same principle.

This principle will be most easily comprehended, if the reader makes a rude model in the following manner:—

Let two slits be cut at right angles to each other, A B, C D, in a rectangular piece of card. Describe a circle on a drawing board, stick one pin upright in the centre E, and another in any point within the circle, as F; put the card down so that one pin may pass through each slit. Then if the card be turned round, so that one edge shall always be tangential to the circle, while the card slides freely on the pins, it will be perceived that a pencil, if held steadily above any point of the card in the line E F, would describe an ellipse on it, the major axis of which would coincide with the diameter through E, F; and the difference of the two semi-axes would be equal to E F.



Since the slits are at right angles, and always pass through F and E, the point G in which they intersect must describe a circle, as is obvious from the figure: and as the side of the card is always to be tangential to the circle, the distance F I varies continually, as the card moves round, being greatest and least when that side is perpendicular to the diameter through E, F; or when the tangent point, I, is at the two vertices of that diameter, in which positions C D will coincide with the diameter perpendicular to E F. When the side of the card is parallel to E F, G will be at F; and C D will coincide with the diameter E F.

It is, on obvious mechanical principles, immaterial to the action of the trammel, whether the bar move round carrying the pencil, and causing the pegs to move in the grooves, while these remain at rest; or whether the bar with the pegs and pencil are fixed, and the cross move.

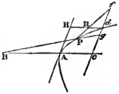
If the instrument were turned upside down, the bar being fixed, and a plate, or *table*, were screwed at the back of the cross; then if the pencil on the bar were contrived so as to stand fixed over any part of the plate, it would describe an ellipse on the plate if this were moved round: this modification is represented by the card, the slits being analogous to the grooves. The instruments above mentioned are contrived on the principle of this modified trammel.

Now if one of the pegs on the bar be supposed to increase till it becomes a ring of considerable diameter, with respect to the plate on the cross—the groove widening with it so as to cause the arm of the cross to assume the form of two bars fixed edgeways at the back of the plate—the friction during the turning round of the plate would be diminished, and the motion consequently would be easier and steadier.

An arbor, on which the plate turns, is substituted for the other peg F; and in order to admit of the distance F I from this peg to the point where the edge-bars touch the ring, varying as was explained above of the piece of card, the arbor F is fixed to a second plate, which slides in a dove-tailed groove cut in the principal plate which carries the edge-bars. By this contrivance the edge-bars move round, clipping the ring, which, as well as the arbor F, are stationary; and the principal, or upper plate, moves backward and forward on the dove-tailed plate which turns round on the arbor. The lathe-chuck, and the elliptograph alluded to, are contrived so as to admit of the ring, or its centre E, being moved to any distance from F, less than the inner semi-diameter of the ring; and when adjusted by trial so that the pencil or *cutter* of the lathe shall produce the required ellipse, E is fixed so as to keep E F at a constant distance. When the ring is moved so that E and F coincide, the pencil will describe a circle. In the chuck, the arbor F is in a line with that of the lathe, the chuck being made so as to put on that arbor; and the substance to be turned elliptically is fixed on the face of the plate, and is thus presented to the tool.

NOTE KK, p. 100.

1. If A and B are the vertices of any diameter of an hyperbola, then lines drawn from A and B through any point P in the curve, will cut any ordinate to A B, as cd, in points f and g, which will divide cg in continued proportion, that is, $cg : cd :: cd : cf$.



(Demonstrated precisely as Lem. 1, Note EE.)

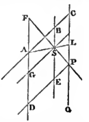
2. If an ordinate d H be drawn any where parallel to A B to cut the tangent at A in H, then lines Af, Bg drawn through any point P in the curve, will cut the ordinates d H, cd in the same ratio, that is $cg : cd :: \text{H R} : \text{H}d$.

For by preceding proposition, $cg : cd :: cd : cf$; and by similar triangles $\text{H R} : \text{R}d :: \text{H A} : df :: cd : df$ (because the tangent $\text{A H} = cd$): and $\text{H R} + \text{R}d\ (\text{H}d) : \text{R}d :: cd + df\ (cf) : df$, therefore $\text{H R} : \text{H}d :: cd : cf$. But $cg : cd :: cd : cf$, consequently $\text{H R} : \text{H}d :: cg : cd$. Q. E. D.

Hence the construction in the text; for by dividing the sides of the parallelogram H$dc$A into an equal number of parts, the proportion between the segments is always preserved.

NOTE LL, p. 105.

Let A D P Q be any diameters of a parabola; through P draw an ordinate to A D, parallel to the tangent A C at the vertex A: then if a line A L be drawn from A through any point S in the curve, to cut Q P produced, in L, and another diameter be drawn through S, to cut the ordinate D P in E: then $\text{C L} : \text{C P} :: \text{D E} : \text{D P}$.



For the diameters A D, S E, P Q being parallel, the segment A C of the tangent, cut off from A by Q P, will be equal to D P the semi-ordinate, and A D will be equal to P C. Draw an ordinate G S parallel to D P or A C, to cut A D in G, and draw P S to cut D A produced in F. Then by the property of the curve $\text{A G} : \text{A D} :: \text{G S}^2 : \text{D P}^2$, that is $\text{B S} : \text{C P} :: \text{A B}^2 : \text{A C}^2$; but by similar triangles $\text{B S}^2 : \text{C L}^2 :: \text{A B}^2 : \text{A C}^2$; therefore, $\text{B S} : \text{C P} :: \text{B S}^2 : \text{C L}^2$, consequently, B S, C L, and C P, are in continued proportion (Eucl. 6, Prop. 20, Cor. 2), or $\text{B S} : \text{C L} :: \text{C L} : \text{C P}$. But $\text{B S} : \text{C L} :: \text{A B} : \text{A C}$; therefore, $\text{C L} : \text{C P} :: \text{A B}\ (\text{D E}) : \text{A C}\ (\text{D P})$. Q. E. D.

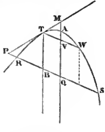
Hence the construction in the text, the division of the sides of the parallelogram into the same *number* of equal parts, preserving the proportion between the segments.

It follows as a corollary to the preceding demonstration, that $AF = CL$. For since $BS : CL :: CL : CP$; and by similar triangles $BS : CL :: AS : AL$ and $AB : AC :: AS : AL$; $AB : AC :: CL : CP$. Again, by similar triangles $FA : LP :: AS : SL$ and $FA : FA + LP :: AS : AS + SL\ (AL)$: and $CL : CP :: AS : AL$ or $CL : CL + LP :: FA : FA + LP$; or $CL : LP :: FA : LP$; that is, $CL = FA$. Q. E. D.

Hence the construction in Case 3rd, p. 105.

Note MM, p. 108.

If RS be an ordinate to any diameter AQ, and TB be another diameter cutting RS in B, then a tangent at T will cut SR produced in P, so that PR, PB, PS, will be in continued proportion.



1. Let p be the parameter to AQ; then $p\,AV = TV^2$ and $p\,AQ = QR^2 = QS^2$ (Lesl. G. C. Prop.): then by subtracting, $p\,AQ - p\,AV = QR^2 - TV^2 = (QR + QB) \cdot (QR - QB)$ because $BQ = TV = VW$; that is $p\,VQ = BS \cdot BR = p\,TB$.

2. Since $MV = 2\,AV$, by similar triangles we have $2\,AV : TV :: TB : PB :: p\,TB \cdot p\,PB$, therefore $p\,TB \cdot TV = p\,PB \cdot 2\,AV = 2\,TV^2 \cdot PB$ (because $p\,AV = TV^2$); and $p\,TB = 2\,TV \cdot PB$ by dividing by TV. Therefore (1) $BS \cdot BR = 2\,TV \cdot PB$: whence $PB : BR :: BS : 2\,TV\ (TW)$; and by division $PB - BR\ (PR) : BR :: BS - TW\ (BR) : TW$, by adding $PR : PR + BR\ (PB) :: BR : BR + TW\ (BS)$, and by adding again $PR : PB :: PR + BR\ (PB) : PB + BS\ (PS)$, or $PR \cdot PS = PB^2$. Q. E. D.

BIBLIOTECA NAZIONALE VITTORIO EMANUELE III
NAPOLI

THE END.

London: Printed by William Clowes, Duke Street, Lambeth.

VA1 1523616

BLIOTECA